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THE TRANSVERSALITY OF ELECTROMAGNETIC WAVES

By J. HODGKINSON

[Received 6 November 1935]

THE doctrine that electromagnetic waves are strictly transverse, i.e., that the electric and magnetic vectors, **E** and **H**, are perpendicular to the direction of propagation, still persists. It should have perished at least as far back as 1889, when Hertz gave the solution of Maxwell's field equations which represents the wave-system set up by an oscillating doublet.

There is a strong temptation to identify the direction of Poynting's vector $\mathbf{E} \wedge \mathbf{H}$ with the direction of propagation. If this may be done, there is, of course, no more to be said on the subject of transversality. If, however, we consider this in relation to Huyghens's principle, it appears that, at every point, the direction of Poynting's vector should coincide with the normal to the wave-front which passes through that point.

In a recent paper* I noted that I was unaware of any suitable mathematical description of a three-dimensional system of progressive waves. I shall repeat the device I employed in that paper and avoid the necessity of finding one by discussing a slightly wider problem. As the successive wave-fronts must be parallel surfaces, I shall substitute parallel surfaces for wave-fronts, and shall have no necessity to inquire whether the wave-systems are progressive or not. I shall show in three special examples the impossibility of Poynting's vector coinciding in direction with the normals to a system of parallel surfaces. The implication of this appears to be that the identification of the direction of Poynting's vector with that of propagation is inconsistent with the doctrine of rectilinear propagation.

THEOREM I. *Given a system of parallel cylindrical surfaces other than circular cylinders or parallel planes, there is no solution of the field equations representing a system of oscillations in which, at every point, Poynting's vector is normal to the surface through the point.*

The position of a point is determined by z its distance from a fixed cross-section of the cylinders, ν its normal distance from a selected

* J. Hodgkinson, *Quart. J. of Math.* (Oxford) 6 (1935), 316–20.

cylinder, and ψ the angle between the tangent plane at the foot of this normal distance and a fixed plane parallel to the generators. If we take the plane $x = 0$ to be this fixed plane, the cartesian coordinates of the point are

$$x+\nu\sin\psi, \qquad y-\nu\cos\psi, \qquad z,$$

where (x, y, z) is the foot of the normal ν, so that*

$$ds^2 = dz^2+(\rho+\nu)^2d\psi^2+d\nu^2.$$

Thus $\dfrac{1}{h_1} = 1, \qquad \dfrac{1}{h_2} = \rho+\nu, \qquad \dfrac{1}{h_3} = 1,$ in a usual notation.

Now, if E_1, E_2, E_3, H_1, H_2, H_3 are the components of the electric and magnetic vectors, the field equations are

$$\frac{1}{c}\frac{\partial}{\partial t}(E_1+iH_1) = ih_2h_3\left[\frac{\partial}{\partial\psi}\left\{\frac{(E_3+iH_3)}{h_3}\right\}-\frac{\partial}{\partial\nu}\left\{\frac{E_2+iH_2}{h_2}\right\}\right], \text{ etc.}$$

If then $E_3 = 0$, $H_3 = 0$, we have

$$\frac{1}{c}\frac{\partial}{\partial t}(E_1+iH_1) = -\frac{i}{\rho+\nu}\frac{\partial}{\partial\nu}\{(\rho+\nu)(E_2+iH_2)\}, \tag{1}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(E_2+iH_2) = i\frac{\partial}{\partial\nu}(E_1+iH_1), \tag{2}$$

$$0 = \frac{\partial}{\partial z}\{(\rho+\nu)(E_2+iH_2)\}-\frac{\partial}{\partial\psi}(E_1+iH_1). \tag{3}$$

From (3) we may write

$$E_1+iH_1 = \frac{\partial U}{\partial z}, \qquad (\rho+\nu)(E_2+iH_2) = \frac{\partial U}{\partial\psi},$$

and then (1), (2) give us

$$\frac{1}{c}\frac{\partial^2U}{\partial z\partial t} = -\frac{i}{(\rho+\nu)}\frac{\partial^2U}{\partial\psi\partial\nu}, \qquad \frac{1}{c(\rho+\nu)}\frac{\partial^2U}{\partial\psi\partial t} = i\frac{\partial^2U}{\partial z\partial\nu}.$$

Hence $$\frac{\partial}{\partial z}\left\{(\rho+\nu)\frac{\partial^2U}{\partial z\partial t}\right\}+\frac{\partial}{\partial\psi}\left(\frac{1}{\rho+\nu}\frac{\partial^2U}{\partial\psi\partial t}\right) = 0,$$

and $$\frac{\partial}{\partial z}\left\{(\rho+\nu)\frac{\partial^2U}{\partial z\partial\nu}\right\}+\frac{\partial}{\partial\psi}\left(\frac{1}{\rho+\nu}\frac{\partial^2U}{\partial\psi\partial\nu}\right) = 0.$$

Write $\dfrac{\partial U}{\partial t} = V$ and we deduce, after differentiating the latter with respect to t,

$$\frac{\partial^2V}{\partial z^2}+\frac{1}{\rho+\nu}\frac{\partial}{\partial\psi}\left(\frac{1}{\rho+\nu}\frac{\partial V}{\partial\psi}\right) = 0, \tag{4}$$

* ρ denotes the radius of curvature of the cross-section at the foot of the normal ν.

$$\frac{\partial^3 V}{\partial z^2 \partial \nu} + \frac{1}{\rho+\nu}\frac{\partial}{\partial \psi}\left(\frac{1}{\rho+\nu}\frac{\partial^2 V}{\partial \psi \partial \nu}\right) = 0. \tag{5}$$

If we differentiate (4) with respect to ν, and then subtract (5), we obtain

$$\frac{2}{(\rho+\nu)^3}\frac{\partial^2 V}{\partial \psi^2} - \frac{3}{(\rho+\nu)^4}\frac{d\rho}{d\psi}\frac{\partial V}{\partial \psi} = 0,$$

whence

$$\frac{\partial V}{\partial \psi}(\rho+\nu)^{-\frac{3}{2}} = W, \tag{6}$$

where W is a function of z, ν, t. Hence from (4)

$$\begin{aligned}\frac{\partial^2 V}{\partial z^2} &= -\frac{W}{\rho+\nu}\frac{\partial}{\partial \psi}\{(\rho+\nu)^{\frac{1}{2}}\} \\ &= -\frac{1}{2}\frac{W}{(\rho+\nu)^{\frac{1}{2}}}\frac{d\rho}{d\psi},\end{aligned} \tag{7}$$

and now, from (6), (7),

$$(\rho+\nu)^{\frac{1}{2}}\frac{\partial^2 W}{\partial z^2} = -\tfrac{1}{2}W\frac{\partial}{\partial \psi}\left\{\frac{1}{(\rho+\nu)^{\frac{1}{2}}}\frac{d\rho}{d\psi}\right\},$$

i.e.

$$\frac{2}{W}\frac{\partial^2 W}{\partial z^2} = -\frac{1}{(\rho+\nu)^{\frac{1}{2}}}\frac{\partial}{\partial \psi}\left\{\frac{1}{(\rho+\nu)^{\frac{1}{2}}}\frac{d\rho}{d\psi}\right\}.$$

Here the left-hand member is a function of z, ν, t, and the right-hand member a function of ψ, ν. Hence we may write each equal to a function of ν only, say $-\frac{1}{2}N$,

i.e.

$$\frac{1}{(\rho+\nu)^{\frac{1}{2}}}\frac{\partial}{\partial \psi}\left\{\frac{1}{(\rho+\nu)^{\frac{1}{2}}}\frac{d\rho}{d\psi}\right\} = \tfrac{1}{2}N,$$

whence

$$\frac{1}{(\rho+\nu)^3}\left(\frac{d\rho}{d\psi}\right)^2 = N\rho+N', \tag{8}$$

where N' is also a function of ν. Hence

$$\frac{\partial}{\partial \nu}\{(\rho+\nu)^3(N\rho+N')\} = 0,$$

i.e.

$$3(N\rho+N')+(\rho+\nu)\left(\rho\frac{dN}{d\nu}+\frac{dN'}{d\nu}\right) = 0.$$

From this

$$\frac{dN}{d\nu} = 0, \qquad 3N+\nu\frac{dN}{d\nu}+\frac{dN'}{d\nu} = 0, \qquad 3N'+\nu\frac{dN'}{d\nu} = 0.$$

These can only be satisfied by $N = 0$, $N' = 0$, whence, from (8),

$$\frac{d\rho}{d\psi} = 0,$$

i.e. the cylinders must be circular.

The case of parallel planes was automatically excluded from consideration by our choice of a coordinate-system.

THEOREM II. *Given a system of parallel surfaces of revolution other than spheres, circular cylinders or planes, there is no solution of the field equations representing a system of oscillations in which, at every point, Poynting's vector is normal to the surface through the point.*

The position of a point is determined by ν its normal distance from a selected surface of the system, ψ the angle between the normal and the axis of revolution, and ϕ the azimuth of the point reckoned from a fixed meridional plane. Then, if we write the cylindrical-polar coordinates of the foot of the normal as z, ϖ, ϕ, the coordinates of the point are

$$z-\nu\cos\psi, \qquad \varpi+\nu\sin\psi, \qquad \phi,$$

so that* $$ds^2 = (\rho+\nu)^2 d\psi^2 + (\varpi+\nu\sin\psi)^2 d\phi^2 + d\nu^2,$$

and $$\frac{1}{h_1} = \rho+\nu, \qquad \frac{1}{h_2} = \varpi+\nu\sin\psi, \qquad \frac{1}{h_3} = 1.$$

If as before we put $E_3 = 0$, $H_3 = 0$, we have

$$\frac{1}{c}\frac{\partial}{\partial t}(E_1+iH_1) = -\frac{i}{(\varpi+\nu\sin\psi)}\frac{\partial}{\partial\nu}\{(\varpi+\nu\sin\psi)(E_2+iH_2)\},$$

$$\frac{1}{c}\frac{\partial}{\partial t}(E_2+iH_2) = \frac{i}{\rho+\nu}\frac{\partial}{\partial\nu}\{(\rho+\nu)(E_1+iH_1)\},$$

$$0 = \frac{\partial}{\partial\psi}\{(\varpi+\nu\sin\psi)(E_2+iH_2)\} - \frac{\partial}{\partial\phi}\{(\rho+\nu)(E_1+iH_1)\}.$$

Thus, writing

$$(\rho+\nu)(E_1+iH_1) = \frac{\partial U}{\partial\psi}, \qquad (\varpi+\nu\sin\psi)(E_2+iH_2) = \frac{\partial U}{\partial\phi}, \tag{9}$$

we have

$$\left.\begin{aligned} \frac{1}{c(\rho+\nu)}\frac{\partial^2 U}{\partial\psi\partial t} &= -\frac{i}{\varpi+\nu\sin\psi}\frac{\partial^2 U}{\partial\phi\partial\nu}, \\ \frac{1}{c(\varpi+\nu\sin\psi)}\frac{\partial^2 U}{\partial\phi\partial t} &= \frac{i}{\rho+\nu}\frac{\partial^2 U}{\partial\psi\partial\nu}, \end{aligned}\right\} \tag{10}$$

whence

$$\frac{\partial}{\partial\psi}\left(\frac{\varpi+\nu\sin\psi}{\rho+\nu}\frac{\partial^2 U}{\partial\psi\partial t}\right) + \frac{\partial}{\partial\phi}\left(\frac{\rho+\nu}{\varpi+\nu\sin\psi}\frac{\partial^2 U}{\partial\phi\partial t}\right) = 0,$$

$$\frac{\partial}{\partial\psi}\left(\frac{\varpi+\nu\sin\psi}{\rho+\nu}\frac{\partial^2 U}{\partial\psi\partial\nu}\right) + \frac{\partial}{\partial\phi}\left(\frac{\rho+\nu}{\varpi+\nu\sin\psi}\frac{\partial^2 U}{\partial\phi\partial\nu}\right) = 0.$$

* Here ρ denotes the radius of curvature of the meridional section at the foot of the normal ν.

After writing $\dfrac{\partial U}{\partial t} = V, \qquad \dfrac{\varpi + \nu \sin\psi}{\rho + \nu} = \chi,$

we have

$$\chi \frac{\partial}{\partial \psi}\left(\chi \frac{\partial V}{\partial \psi}\right) + \frac{\partial^2 V}{\partial \phi^2} = 0, \tag{11}$$

$$\chi \frac{\partial}{\partial \psi}\left(\chi \frac{\partial^2 V}{\partial \psi \partial \nu}\right) + \frac{\partial^3 V}{\partial \phi^2 \partial \nu} = 0. \tag{12}$$

Hence $$2\chi \frac{\partial \chi}{\partial \nu}\frac{\partial^2 V}{\partial \psi^2} + \left(\frac{\partial \chi}{\partial \nu}\frac{\partial \chi}{\partial \psi} + \chi \frac{\partial^2 \chi}{\partial \psi \partial \nu}\right)\frac{\partial V}{\partial \psi} = 0,$$

which leads to $$\frac{\partial V}{\partial \psi}\left(\chi \frac{\partial \chi}{\partial \nu}\right)^{\frac{1}{2}} = W, \tag{13}$$

a function of ϕ, ν, t and now, from (11)

$$\frac{\partial^2 V}{\partial \phi^2} = -W\chi \frac{\partial}{\partial \psi}\left(\chi \Big/ \frac{\partial \chi}{\partial \nu}\right)^{\frac{1}{2}}.$$

This, with (13), now gives

$$\frac{1}{W}\frac{\partial^2 W}{\partial \phi^2} = -\left(\chi \frac{\partial \chi}{\partial \nu}\right)^{\frac{1}{2}} \frac{\partial}{\partial \psi}\left\{\chi \frac{\partial}{\partial \psi}\left(\chi \Big/ \frac{\partial \chi}{\partial \nu}\right)^{\frac{1}{2}}\right\}.$$

Thus, as in the preceding theorem, we may write

$$\left(\chi \frac{\partial \chi}{\partial \nu}\right)^{\frac{1}{2}} \frac{\partial}{\partial \psi}\left\{\chi \frac{\partial}{\partial \psi}\left(\chi \Big/ \frac{\partial \chi}{\partial \nu}\right)^{\frac{1}{2}}\right\} = \tfrac{1}{4}N,$$

where N is a function of ν only;

i.e. $$\left(\chi \frac{\partial \chi}{\partial \nu}\right)^{\frac{1}{2}} \frac{\partial}{\partial \psi}\left\{\left(\chi \frac{\partial \chi}{\partial \nu}\right)^{\frac{1}{2}} \frac{\partial}{\partial \psi}\left(\chi \Big/ \frac{\partial \chi}{\partial \nu}\right)\right\} = \tfrac{1}{2}N,$$

i.e. $$2\chi \frac{\partial \chi}{\partial \nu}\frac{\partial^2}{\partial \psi^2}\left(\chi \Big/ \frac{\partial \chi}{\partial \nu}\right) + \frac{\partial}{\partial \psi}\left(\chi \frac{\partial \chi}{\partial \nu}\right)\frac{\partial}{\partial \psi}\left(\chi \Big/ \frac{\partial \chi}{\partial \nu}\right) = N.$$

Thus $$\chi \frac{\partial \chi}{\partial \nu}\left\{\frac{\partial}{\partial \psi}\left(\chi \Big/ \frac{\partial \chi}{\partial \nu}\right)\right\}^2 = N\left(\chi \Big/ \frac{\partial \chi}{\partial \nu}\right) + N',$$

where N' is also a function of ν. Replacing χ by its value, we have

$$\left\{\frac{(\rho+\nu)^2 \cos\psi}{\rho \sin\psi - \varpi} - \frac{d\rho}{d\psi}\frac{(\varpi + \nu\sin\psi)^2}{(\rho\sin\psi - \varpi)^2}\right\}^2$$
$$= \frac{N(\rho+\nu)^4}{(\rho\sin\psi - \varpi)^2} + \frac{N'(\rho+\nu)^3}{(\rho\sin\psi - \varpi)(\varpi + \nu\sin\psi)}.$$

Since this is true for all values of ψ, ν, we find on putting $\nu = -\rho$ that $d\rho/d\psi$ must be zero, i.e. the surfaces are spheres.

The other excepted cases were automatically excluded from consideration by our choice of a coordinate-system.

THEOREM III. *Given a solution of the field equations representing a system of oscillations in which, at every point, Poynting's vector is directed along the radius from a fixed point, there must be some singularity of the field on each sphere which has the fixed point as centre.*

In the work of the preceding section we replace ν by r, ρ by 0, ψ by the polar angle θ, χ by $\sin\theta$; then (12), in its new form, is immediately deducible from (11), which is now

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{\partial^2 V}{\partial\phi^2}=0.$$

We put $\qquad u=\tan\tfrac{1}{2}\theta e^{i\phi}, \qquad v=\tan\tfrac{1}{2}\theta e^{-i\phi},$

and obtain

$$\frac{\partial^2 V}{\partial u\partial v}=0,$$

i.e. we may write

$$V=\frac{\partial}{\partial t}\{F_1(u,r,t)+F_2(v,r,t)\},$$

and thence

$$U=F_1(u,r,t)+F_2(v,r,t).$$

Substitution in (10) leads to

$$\frac{1}{c}\frac{\partial^2 F_1}{\partial u\partial t}=\frac{\partial^2 F_1}{\partial u\partial r}, \qquad \frac{1}{c}\frac{\partial^2 F}{\partial v\partial t}=-\frac{\partial^2 F}{\partial v\partial r},$$

so that we may put

$$U=G_1(u,ct+r)+G_2(v,ct-r),$$

and now, from (9),

$$E_1+iH_1=\frac{1}{r\sin\theta}\left(u\frac{\partial G_1}{\partial u}+v\frac{\partial G_2}{\partial v}\right),$$

$$E_2+iH_2=\frac{i}{r\sin\theta}\left(u\frac{\partial G_1}{\partial u}-v\frac{\partial G_2}{\partial v}\right).$$

We seek forms of G_1, G_2 so that E_1, H_1, E_2, H_2 shall be finite for all values of θ, and all values of r except $r=0$.

It is necessary that

$$\lim\left(\frac{1}{\sin\theta}\,u\frac{\partial G_1}{\partial u}\right) \quad \text{and} \quad \lim\left(\frac{1}{\sin\theta}\,v\frac{\partial G_2}{\partial v}\right)$$

shall exist when θ tends either to 0 or to π, in order that E_1, H_1, E_2, H_2 may be finite for these values of θ. If we project the surface of the sphere of radius r stereographically upon the tangent plane at the point $\theta=0$, with vertex of projection at the point $\theta=\pi$, the cartesian coordinates ξ, η of a point in that plane are connected with

the angular coordinates θ, ϕ of the corresponding point on the sphere by means of the relations

$$\zeta \equiv \xi + i\eta = 2r \tan \tfrac{1}{2}\theta e^{i\phi} = 2ru.$$

Writing $u\dfrac{\partial G_1}{\partial u} = K(u, ct+r)$, we require that $K(\zeta/2r, ct+r)$ shall be without singularities, *qua* function of ζ, except possibly at the values zero and infinity, and further that

$$\lim\{|\zeta| + |\zeta|^{-1}\}K(\zeta/2r, ct+r)$$

shall exist as ζ tends either to zero or infinity. This plainly will not happen if zero (or infinity) is an ordinary point (not a zero) of the function, nor if it is a pole. If it is an essential singularity, the limit of the function K itself can be made infinite by a proper approach to zero (or infinity); much more so the limit required. Hence the two points, zero and infinity, must be zeros of the function K. Thus K is a function of ζ without singularities and taking the value 0 at zero and infinity. Thus K is itself zero.

We may treat $v \operatorname{cosec} \theta \, \partial G_2/\partial v$ in a similar way, and thus obtain the result that, if there is no singularity, E_1, H_1, E_2, H_2 are all zero, i.e. there is no field.

Thus we conclude that, if, in fact, the strict transversality of electromagnetic waves were necessary, either it would be impossible for complete spherical waves to exist, or, perhaps, there would have to be surfaces of discontinuity of **E**, **H** in the field, comparable in a way to the surfaces of discontinuity of hydrodynamics. Such a surface would stretch out from the centre, but would not necessarily be conical. There may be no insuperable mathematical difficulty in formulating conditions for such surfaces. I have, in fact, made a partial inquiry into their nature, but they seem to imply that the surface of discontinuity would be the seat of a displacement current-sheet, and the notion seems singularly far-fetched.

It is perhaps advisable, since the phrase 'surface of discontinuity' has been used, to point out that the fantastic conception just described has no connexion with the conception of a moving surface of discontinuity separating an existing field from one in process of establishment, a natural conception which has been adequately discussed by Love.*

* A. E. H. Love, *Proc. London Math. Soc.* (2), 1 (1903), 37–62; *Proc. Roy. Soc.* 74 (1904), 73–83.

ON A QUARTIC PRIMAL WITH FORTY-FIVE NODES, IN SPACE OF FOUR DIMENSIONS

By J. A. TODD (*Manchester*)

[Received 29 January 1936]

1. Introduction

In the course of his investigations on the trisection of the periods of quadruply-periodic hyperelliptic functions Burkhardt† was led to a representation of the simple group of order 25920 as a collineation group in complex projective space of four dimensions, leaving invariant a figure of 45 points lying by nines in 40 planes. In 1906 Coble‡ proved that these points were the double points of a quartic primal which was left unchanged by all the collineations of the group, and gave explicit forms for the equation of the primal and the coordinates of its nodes. Using analytic methods Coble§ developed the properties of this primal and the configuration formed by its nodes, and established a correspondence between these nodes and the tritangent planes of a cubic surface. In those papers no mention is made of the fact that the primal in question is rational, and admits of a simple representation on ordinary space. In this paper I show that this is in fact the case, and starting from the representation I derive the principal properties of the figure by methods which are purely synthetic, and which seem to be essentially simpler than those employed by Coble. In the course of the work I show that the simple group of order 25920 may be generated by two operations of period five, whose product is of period three, these three periods being smaller than those given by other known pairs of generators.

The primal discussed here has another claim to interest besides its group of collineations, for it is probable that it has the greatest number of nodes possible for its order. In fact, if we assume that the prescription of k nodes for a primal reduces its constant number by k, then the freedom of a quartic primal with more than 45 nodes would be less than 69—45, i.e. 24; and so such a primal would admit of an infinite group of collineations. But it is easily seen that no figure with so many as 45 points can be invariant under an infinite group without so many of the points lying in linear spaces as to force the quartic primal having nodes at these points to have multiple

† Burkhardt, **1**. ‡ Coble, **2**. § Coble, **2, 3**.

curves or surfaces. Thus it seems likely that 45 is the greatest number of nodes that a quartic primal in [4] can possess.

2. A figure in [3]

Let A, B, C, D, E be five points in a space Σ of three dimensions, no four of which are coplanar. Consider the problem of determining points $\alpha, \beta, \gamma, \delta, \epsilon$ lying respectively on CD, DE, EA, AB, BC, and such that the lines $A\alpha, B\beta, C\gamma, D\delta, E\epsilon$, in the order named, form a closed pentagon. If a point α_1 is given on CD, the transversal from B to $A\alpha_1$ and DE will meet DE in a point β_1, the transversal from C to $B\beta_1$ and EA will meet EA in a point γ_1, and similarly points δ_1, ϵ_1, are determined on AB and BC. The transversal from A to $E\epsilon_1$ and CD will meet CD in a point α_2. Between the points α_1 and α_2 there is a one-to-one correspondence, and there will be two positions of α_1 on CD for which α_1 and α_2 coincide. If α_1 is chosen to be either of these points, the pentagon $A\alpha_1, B\beta_1, C\gamma_1, D\delta_1, E\epsilon_1$ closes.

There are therefore two solutions to the problem. Denote by a, b, c, d, e the vertices of one of the inscribed pentagons, where cd passes through A, de through B, and so on, and let accents refer to the other pentagon. The projectivity which permutes A, B, C, D, E in cyclic order must leave each inscribed pentagon unchanged, since its period is odd. Hence the ranges $(Acd\alpha)$ and $(Bde\beta)$ on $A\alpha$, $B\beta$, respectively, are projective.

The lines Ae, cB, $\alpha\beta$, all meet in a point of CE, for they lie in the plane cde and lie with CE in the respective planes ACE, BCE, DCE. Hence the range $(dAc\alpha)$ on $A\alpha$ is projective with $(deB\beta)$ on $B\beta$, and hence with $(cdA\alpha)$. This range is therefore *equianharmonic*.

The collineation which keeps A fixed and interchanges B with E and C with D must interchange the pentagons $abcde$ and $a'b'c'd'e'$, for, if $abcde$ were left unaltered, the range $(Bed\beta)$ would be projective with $(Ebc\epsilon)$ and therefore with $(Bde\beta)$; this would therefore be a harmonic, and not an equianharmonic range. It follows that $(Bed\beta)$ is projective with $(Bd'e'\beta')$.

The figure evidently involves no projective invariants, any two such figures being projective.

Henceforth we consider one definite inscribed pentagon $abcde$, and the figure formed by this and the lines AB, BC, CD, DE, EA. This figure admits a cyclic group of collineations permuting A, B, C, D, E in order, but no others.

3. Consider now the linear system $|F|$ of quartic surfaces passing through the ten lines AB, $A\alpha$, etc. There are ∞^4 of these, all of which have double points at A, B, C, D, E, since three non-coplanar lines of the surfaces pass through each of these points. The residual intersection of two of the surfaces is a sextic curve of genus three passing through A, B, C, D, E and meeting each of the ten lines in one further point. Three of the surfaces have thus four variable intersections. Thus the system $|F|$ maps the prime sections of a rational quartic primal V of [4], and, since the ten lines together form a degenerate curve of virtual genus eleven, it follows that V is determinantal.

4. Nodes and planes on V

We now consider the representation on V of the elements of the base-curves of $|F|$. To the neighbourhoods of points on the line AB correspond lines of V forming a pencil. The plane **AB** of this pencil lies on V, its vertex AB is a node. Similarly the points of $D\delta$ correspond to lines in a plane **D**δ passing through a node Dδ. The planes **AB**, **D**δ, both contain the nodes AB, Dδ, the join of which contains a third node δ^* mapped on the neighbourhood of δ, and similarly the neighbourhood of a maps a node, a, lying on the join of Cγ, Dδ. The neighbourhood of A is mapped on a plane **A** on V in such a way that the points of a line of **A** correspond to points in the neighbourhood of A lying on a quadric cone through AB, AE, $A\alpha$ (the tangent cone to a surface F at A). Thus the points of the neighbourhood of A lying in either of the planes ABE, $AB\alpha$, $AE\alpha$ are exceptional and represent nodes A_{BE}, $A_{B\alpha}$, $A_{E\alpha}$ of V lying in **A**, and the points near A which lie in a general plane through A are mapped by a conic passing through these three points.

The diagonals AC, BD, etc., of the pentagon $ABCDE$ are fundamental lines for $|F|$ and map nodes AC, BD, etc., on V; the transversal from A to BC and DE is also fundamental and maps a node A*. In all, V has 45 nodes typified by AB, Aα, a, α^*, A_{BE}, $A_{B\alpha}$, $A_{E\alpha}$, AC, A*.

The planes ABC, ACD, abc represent planes **ABC**, **ACD**, **abc** on V, since the residual intersection of $|F|$ with each of them is a homaloidal net. The quadric surface defined by the lines $A\alpha$, BC, DE, which contains also CD, $B\beta$, $E\epsilon$, represents a plane **A*** on V, and the cubic surface containing all the lines except AB maps a plane **AB***. Thus

V has 40 planes lying on it, typified by **A**, **AB**, **A**α, **ABC**, **ACD**, **abc**, **A***, **AB***.

5. The set of points and planes so obtained is represented unsymmetrically in Σ, the elements appearing in sets of five. But, in fact, the figure in [4] is completely symmetrical. We shall prove this by establishing the existence of a collineation in [4] leaving the figure invariant and changing one of the 40 planes into any other one.

In the first place the cyclic collineation of Σ which permutes A, B, C, D, E in order determines a collineation S in [4] leaving V invariant, whose effect on the nodes and planes is that indicated by the cyclic permutation of the letters occurring in their symbols.

Consider now the web of quadric surfaces in Σ passing through the points E, A, B and the line CD. These form a homaloidal system,† and may be mapped on the planes of a second space Σ_1. To the planes of Σ correspond cubic surfaces in Σ_1 passing through three skew lines l_1, l_2, l_3, and having a transversal l of these as a double line. Let the intersections of l with l_1, l_2, l_3 be called E_1, β_1, D_1, respectively. The neighbourhoods of the points E, A, B of Σ correspond to the planes ll_1, ll_2, ll_3, respectively, and the neighbourhood of CD to the quadric surface containing l_1, l_2, l_3, and l. To the points C and D correspond two transversals p, q of l_1, l_2, l_3; let p meet these lines respectively in γ_1, e_1, C_1, and let q meet them in A_1, d_1, α_1. To the lines AE, AB correspond the points D_1, E_1; to the lines DE, BC correspond A_1, C_1. To $A\alpha$ there corresponds a point B_1 of l_2, and to $B\beta$, $E\epsilon$ correspond the lines B_1A_1, B_1C_1. To $D\delta$ corresponds the transversal from E_1 to q and B_1C_1, meeting these in points c_1, ϵ_1; and to $C\gamma$ corresponds the transversal from D_1 to p and A_1B_1, meeting these in a_1, δ_1; since $C\gamma$ and $D\delta$ meet, these two last lines must meet in a point b_1. In Σ_1 we now have a pentagon $A_1B_1C_1D_1E_1$ and points $\alpha_1, \beta_1, \ldots$ on the sides of this such that $A\alpha_1, B\beta_1, \ldots$ form a closed pentagon, i.e. a figure similar in all respects to that in Σ. Further, the range of points $(Cae\gamma)$ on $C\gamma$ is mapped on the range $(a_1b_1\delta_1D_1)$ on $D_1\delta_1$, these ranges are thus projective and hence $(Cae\gamma)$ is projective with $(C_1a_1e_1\gamma_1)$. Hence the collineation which changes A_1, B_1, C_1, D_1, E_1, respectively into A, B, C, D, E changes $a_1b_1c_1d_1e_1$ into $abcde$ (and not into $a'b'c'd'e'$). We have thus a Cremona transformation in Σ which is easily seen to leave $|F|$

† For this transformation see Hudson, **5**, 287.

invariant, and this corresponds to a second collineation T in [4] which leaves V unaltered. It is a straightforward matter to determine from the transformation the effect of T on the nodes and planes of V. We find that T permutes the 40 planes as follows:

(A, BDE, cde, EAB, DE) (B, CDE, EA, D, Aα) (C, Cγ, Dδ, Eϵ, BC) (E, DEA, ACD, Bβ, AB) (CD, B*, DAB, bcd, BCD) (ABC, CEA, eab, abc, A*) (EBC, dea, C*, BC*, E*) (D*, AB*, EA*, CD*, DE*).

The effect of T on the 45 nodes is the following:

(a, b, ϵ*, C_{DB}, $C_{D\gamma}$) (c, $B_{C\beta}$, C*, AE, $D_{C\delta}$) (d, $B_{A\beta}$, E*, E_{AD}, AD) (e, δ*, $E_{D\epsilon}$, A*, AC) (α*, B*, D_{EC}, $A_{E\alpha}$, BD) (β*, A_{BE}, DE, $A_{B\alpha}$, BE) (γ*, $D_{E\delta}$, Aα, B_{CA}, CE) (D*, CD, Bβ, AB, $E_{A\epsilon}$) (Cγ, Dδ, Eϵ, BC, $C_{B\gamma}$).

It is easily verified that the collineations S and T generate a group which is transitive on the planes and nodes of V, and this establishes the symmetry of the configuration. It can also be verified without difficulty that ST is of period nine, and that ST^{-1} is of period three.

6. It is now a simple matter to show that each plane of V contains nine of the nodes, and to construct the table of incidence. In fact the nodes in **A** are obvious from the representation, and those in the other planes can be obtained from these by means of the collineations S and T. In this manner we obtain the following table, which gives the nine nodes lying in one plane of each of the eight sets obtained above; the nodes lying in the other planes are obtained by cyclic permutation of the letters involved.

A: A_{BE}, $A_{B\alpha}$, $A_{E\alpha}$, AB, AE, Aα, AC, AD, A*.
AB: A_{BE}, B_{CA}, $A_{B\alpha}$, $B_{A\beta}$, AB, Dδ, C*, E*, δ*.
Aα: $A_{B\alpha}$, $A_{E\alpha}$, CD, Aα, Bβ, Eϵ, c, d, α*.
ABC: B_{CA}, $A_{E\alpha}$, $C_{D\gamma}$, DE, AC, A*, C*, δ*, ϵ*.
ACD: A_{BE}, $C_{B\gamma}$, $D_{E\delta}$, CD, AC, AD, BE, c, d.
abc: $D_{E\delta}$, $E_{D\epsilon}$, AC, B*, a, c, β*, δ*, ϵ*.
A*: A_{BE}, D_{EC}, C_{DB}, $B_{C\beta}$, $E_{D\epsilon}$, Aα, A*, b, e.
AB*: $A_{E\alpha}$, $B_{C\beta}$, CD, DE, Cγ, Eϵ, CE, D*, δ*.

The following facts are easily verified from the above table. Each node lies in eight of the planes. These lie on the tangent cone to V at the node, four belonging to each system on the cone. The four lines in which one of these planes is met by the four planes of the opposite

system form an equianharmonic pencil; one such set of lines, in the plane $\mathbf{A}\alpha$, is mapped by the neighbourhoods, on $A\alpha$, of the points A, c, d, α. The nine points in each plane lie by threes on twelve lines, and thus are the inflexions of a pencil of plane cubics. Through each of these lines containing three nodes there passes a second plane of the set, and the three planes determined in this way by three lines in one of the planes which contain between them all nine nodes lie in a prime. There are thus forty primes which meet V in four planes; one such contains the planes **A**, **AB**, **ABC**, **BC***. Four of the primes pass through each plane, and they form an equianharmonic pencil. Each of the primes contains eighteen nodes lying by threes on the six lines of intersection of the four planes in the prime.

These properties establish the identity of the figure with that considered by Burkhardt and Coble.

7. The collineation group leaving V invariant

It is not difficult to show directly that V admits a group of 25920 collineations. In the first place the group of *automorphisms* of the figure of points and nodes (i.e. of permutations of these elements which leave the table of incidence unaltered) is of order 51840; this is most easily seen by noticing that we can set up a correspondence between the 45 nodes of V and the tritangent planes of a cubic surface (whose group is known to have this order) in such a way that the line joining two nodes of V lies in a plane of V, if and only if the line of intersection of the corresponding tritangent planes is not a line of the cubic surface.†

Now consider the figure of § 2, with the two inscribed pentagons $abcde$, $a'b'c'd'e'$. The collineation of Σ which keeps A fixed but interchanges B and E, C and D, transforms $|F|$ into a system of quartic surfaces $|F'|$ passing through AB, $A\alpha'$, etc., and these represent the prime sections of a quartic primal V'. The correspondence between the nodes and planes of V and those of V' represented by corresponding elements in Σ has the properties of an anticollineation, and in particular it changes the cross-ratio of any equianharmonic pencil into its conjugate. Since the figure in [3] has no projective invariants, the same is true of the figure in [4]. Hence any automorphism of the nodes and planes of V can be realized either as a collineation or an anticollineation. Thus the group of collineations is of order 25920,

† Cf. Coble, **2**, 365.

and, being of index two in the larger group of order 51840, it must be the simple group of this order.

It is interesting to note that S and T generate the entire group. For the group generated by S and T is transitive on the forty planes, and since ST is of period nine the order of the group $\{S, T\}$ must be a multiple of 360. On the other hand, the only proper sub-groups of the simple group of order 25920 whose orders are divisible by 360 are simply isomorphic† with the alternating or symmetric groups of degree six, neither of which contain operations of period nine. Hence S and T generate the whole group. Since ST^{-1} is of period three it follows that *the simple group of order* 25920 *may be generated by two operations of period five, whose product is of period three.*

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† Dickson, **4**.

RAMANUJAN'S INTEGRALS AND GAUSS'S SUMS

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1. THE most general of the many results contained in Ramanujan's paper,* *Some definite integrals connected with Gauss's sums*, is that, when n and θ are real with

$$n > 0, \qquad -1 < \theta < 1, \qquad \Re(t) > 0, \qquad |\Im(t)| < \pi,$$

then

$$\sin\pi\theta \int_0^\infty \frac{e^{-i\pi nx^2}\cos tx}{\cosh\pi x + \cos\pi\theta}\,dx$$

$$= \sum_{r=0}^{\infty} \{e^{-(2r+1-\theta)t+(2r+1-\theta)^2 i\pi n} - e^{-(2r+1+\theta)t+(2r+1+\theta)^2 i\pi n}\} +$$

$$+ \frac{1}{\sqrt{n}} \exp\left\{-\frac{i}{4}\left(\pi - \frac{t^2}{\pi n}\right)\right\} \sum_{r=0}^{\infty} (-1)^{r-1} \sin r\pi\theta\, e^{-(2rt+r^2+i\pi)/(4n)}. \quad (1)$$

If we take $\theta = \frac{1}{2}$ and replace t by $2t$, the formula becomes

$$\int_0^\infty e^{-i\pi nx^2} \frac{\cos 2tx}{\cosh\pi x}\,dx = \sum_{r=0}^{\infty} (-1)^r e^{-(2r+1)t+(2r+1)^2 i\pi n/4} +$$

$$+ \frac{1}{\sqrt{n}} \exp\left\{-i\left(\frac{\pi}{4} - \frac{t^2}{\pi n}\right)\right\} \sum_{r=0}^{\infty} (-1)^r e^{-(2r+1)t/n-(2r+1)^2 i\pi/(4n)}, \quad (2)$$

with $$n > 0, \qquad \Re(t) > 0, \qquad |\Im(t)| < \tfrac{1}{2}\pi.$$

If, in addition, we take $n = 1$ and replace t by πt, we obtain the still simpler formula

$$\int_0^\infty e^{-i\pi x^2} \frac{\cos 2\pi tx}{\cosh\pi x}\,dx = \frac{(1+i)/\sqrt{2} - ie^{i\pi t^2}}{2\cosh\pi t}, \quad (3)$$

with $$|\Im(t)| < \tfrac{1}{2},$$

the sign of $\Re(t)$ now being obviously immaterial.

Ramanujan's proof of (2) was based on an application of Lerch's theorem which, like much of his work, makes one wonder how he discovered the result. He did not trouble to write out a formal proof of (1), but merely explained how it could be constructed in the same way as (2).

Subsequently Mordell† obtained various results of this kind as

* *Messenger of Math.* 44 (1915), 75–85.

† *Quart. J. of Math.* 48 (1920), 329–42; *Acta Math.* 61 (1933), 323–60.

limiting cases of formulae of a similar character in which the index in the exponential function under the integral sign was not a pure imaginary. The method used by Mordell was, like Ramanujan's method, indirect; it consisted in proving that the integral under consideration satisfied a pair of functional equations and constructing a series which satisfied the same equations; the proof was then completed by establishing the uniqueness of the solution of the functional equations.

2. For some time I have felt that, in view of its indirect character, Ramanujan's method did not provide a 'best possible' proof of (1). Consequently, when, a few years ago, I had occasion* to make use of (3), I deemed it advisable to construct a direct proof of (3) by contour integration. Unfortunately, my proof, which was less simple than one would have wished, did not lend itself to such an extension as would cover values of n other than 1.

Recently, on happening to look again at Ramanujan's paper, I noticed that it is possible to obtain his results by a very simple and direct application of the calculus of residues; all that is necessary is to write

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{c} + \int_{c}^{c+\infty i} + \int_{c+\infty i}^{c} + \int_{c}^{\infty},$$

where c is a suitably chosen constant, and then to evaluate the sum of the first two integrals on the right as $2\pi i$ times the sum of the residues of the integrand at its poles above and on the left of the path of integration, and to evaluate the sum of the last two integrals on the right by expanding the integrand into a series and integrating term by term.

In this paper I propose to follow the method just explained in order to construct proofs of a pair of formulae, namely (1) and

$$\int_0^{\infty} \frac{e^{-i\pi n x^2} \sin tx \sinh \pi x}{\cosh \pi x + \cos \pi\theta}\, dx$$

$$= \sum_{m=0}^{\infty} \left\{ e^{-(2m+1-\theta)t+(2m+1-\theta)^2 i\pi n} + e^{-(2m+1+\theta)t+(2m+1+\theta)^2 i\pi n} \right\} -$$

$$- \frac{1}{\sqrt{n}} \exp\left(\frac{it^2}{4n\pi} + \frac{i\pi}{4}\right) \sum_{m=0}^{\infty} (-1)^m \tfrac{1}{2}\epsilon_m \cos m\pi\theta\, e^{-(2mt+m^2 i\pi)/(4n)}. \quad (4)$$

* *Compositio Math.* 1 (1935), 63.

Here ϵ_m denotes 'Neumann's factor' which is defined to be 1 when m is zero and to be 2 when m has any integral value other than zero.

My method is equally available for a direct evaluation of the second (but not the first) of the two standard types of integrals which are discussed by Mordell; I give a summary of this investigation as an addendum to the paper.

3. To simplify the work as much as possible, I shall write out the proofs of (1) and (4) on the assumption that θ and t are both real (t being, of course, positive) and I shall then explain the modifications of detail which must be introduced in extending the analysis to prove (1) and (4) for suitably-chosen complex domains of θ and t.

It conduces to brevity of exposition, if it is understood, once for all, that, in such an integral as

$$\int_{a+\infty i}^{\infty+bi} \quad (a \text{ and } b \text{ real})$$

in which the limits of integration are at infinity in different directions, the path of integration is to be interpreted as being two perpendicular straight lines; one of these lines, parallel to the imaginary axis, starts at infinity and is directed downwards so as to terminate at the point $a+bi$, while the other, parallel to the real axis, starts at this point and is directed towards the right so as to terminate at infinity. The object of this convention is twofold; it fixes the position of the contour relative to the poles of some of the integrands (these poles lie on a line parallel to the imaginary axis) and it prevents the contour from straying into what may be described as mountainous regions in the remote parts of the interior of the first and third quadrants.

4. Let

$$n > 0, \quad -1 < \theta < 1, \quad t > 0, \quad 0 < c < \frac{t}{2n\pi},$$

and consider the series of integrals

$$\sum_{m=0}^{\infty} (-1)^m \epsilon_m e^{m\pi i\theta} \int_{c+\infty i}^{\infty} \exp\{-n\pi iz^2 - (m\pi - it)z\}\, dz.$$

It is evident that, when $m \geqslant 1$, we have

$$\int_{c+\infty i}^{\infty} |\exp\{-n\pi iz^2 - (m\pi - it)z\}\, dz\,|$$

$$= \int_{c}^{\infty} \exp(-m\pi x)\, dx + \int_{0}^{\infty} |\exp\{-n\pi i(c+iy)^2 - (m\pi - it)(c+iy)\}|\, dy$$

$$= \int_c^\infty \exp(-m\pi x)\,dx + \int_0^\infty \exp(2n\pi cy - m\pi c - yt)\,dy$$

$$= e^{-m\pi c}\left(\frac{1}{m\pi} + \frac{1}{t-2n\pi c}\right),$$

and so it is clearly permissible to change the order of summation and integration. Hence

$$\sum_{m=0}^{\infty} (-1)^m \epsilon_m e^{m\pi i\theta} \int_{c+\infty i}^{\infty} \exp\{-n\pi iz^2 - (m\pi - it)z\}\,dz$$

$$= \int_{c+\infty i}^{\infty} \frac{1-e^{\pi i\theta - \pi z}}{1+e^{\pi i\theta - \pi z}} \exp(-n\pi iz^2 + itz)\,dz.$$

Now, for any fixed m, we have, by a double application of Jordan's lemma,

$$\int_{c+\infty i}^{\infty} \exp\{-n\pi iz^2 - (m\pi - it)z\}\,dz$$

$$= \int_{\infty \exp \frac{1}{2}\pi i}^{\infty \exp(-\frac{1}{2}\pi i)} \exp\{-n\pi iz^2 - (m\pi - it)z\}\,dz$$

$$= \int_{-\infty}^{\infty} \exp\{-n\pi w^2 - (m\pi - it)we^{-\frac{1}{2}\pi i}\}.e^{-\frac{1}{2}\pi i}\,dw$$

$$= \frac{1}{\sqrt{n}} \exp\left\{-\frac{\pi i}{4} - \frac{i(m\pi - it)^2}{4n\pi}\right\}.$$

We have thus proved that

$$\int_{c+\infty i}^{\infty} \frac{1-e^{\pi i\theta - \pi z}}{1+e^{\pi i\theta - \pi z}} \exp(-n\pi iz^2 + itz)\,dz$$

$$= \frac{1}{\sqrt{n}} \exp\left(\frac{it^2}{4n\pi} - \frac{i\pi}{4}\right) \sum_{m=0}^{\infty} (-1)^m \epsilon_m e^{m\pi i\theta} \exp\left(-\frac{2mt + i\pi m^2}{4n}\right).$$

We now turn our attention to the integral

$$\int_{-\infty}^{c+\infty i} \frac{1-e^{\pi i\theta - \pi z}}{1+e^{\pi i\theta - \pi z}} \exp(-n\pi iz^2 + itz)\,dz.$$

The poles of the integrand above and on the left of the contour now under consideration are the points

$$z = (2m+1+\theta)i \qquad (m = 0, 1, 2, \ldots),$$

and the residue of the integrand at $(2m+1+\theta)i$ is

$$\frac{2}{\pi}\exp\{-t(2m+1+\theta)+n\pi i(2m+1+\theta)^2\}.$$

In order to prove that the integral now under consideration is equal to $2\pi i$ times the sum of these residues, it is sufficient to prove that, if we write

$$z = c+Re^{i\phi},$$

we can make $R\to\infty$ in such a way that

$$\int_{\frac{1}{2}\pi}^{\pi}\frac{1-e^{\pi i\theta-\pi z}}{1+e^{\pi i\theta-\pi z}}\exp(-n\pi iz^2+itz).Re^{i\phi}id\phi\to 0.$$

To effect this purpose we take

$$R = \sqrt{\{c^2+(2m+\theta)^2\}},$$

and make m tend to infinity through integral values; the quadrantal contour then meets the imaginary axis at a point midway between consecutive poles of the integrand, and so, as R tends to infinity through the values specified, the function

$$\frac{1-e^{\pi i\theta-\pi z}}{1+e^{\pi i\theta-\pi z}}$$

remains bounded on the quadrantal contour. It now follows from Jordan's lemma that

$$\lim_{R\to\infty}\int_{\frac{1}{2}\pi}^{\pi}\frac{1-e^{\pi i\theta-\pi z}}{1+e^{\pi i\theta-\pi z}}\exp(-n\pi iz^2+itz)\frac{dz}{d\phi}\,d\phi = 0,$$

and hence that

$$\int_{-\infty}^{c+\infty i}\frac{1-e^{\pi i\theta-\pi z}}{1+e^{\pi i\theta-\pi z}}\exp(-n\pi iz^2+itz)\,dz$$
$$= 4i\sum_{m=0}^{\infty}\exp\{-t(2m+1+\theta)+n\pi i(2m+1+\theta)^2\}.$$

When we combine the two results that have now been obtained, we see that, since

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{c+\infty i}+\int_{c+\infty i}^{\infty}$$

we have

$$\int_{-\infty}^{\infty} \frac{1-e^{\pi i\theta-\pi z}}{1+e^{\pi i\theta-\pi z}} \exp(-n\pi iz^2+itz)\,dz$$

$$= 4i \sum_{m=0}^{\infty} \exp\{-t(2m+1+\theta)+n\pi i(2m+1+\theta)^2\}+$$

$$+\frac{1}{\sqrt{n}} \exp\left(\frac{it^2}{4n\pi}-\frac{i\pi}{4}\right) \sum_{m=0}^{\infty} (-1)^m \epsilon_m e^{m\pi i\theta} \exp\left(-\frac{2mt+i\pi m^2}{4n}\right). \quad (5)$$

Add to (5) the result obtained from it by replacing θ by $-\theta$, and we deduce (4) immediately; (1) is derived in like manner by subtraction.

5. By making slight modifications in the analysis of § 4, we can establish (1) and (4) when θ is a complex number such that

$$|\Re(\theta)| < 1, \qquad t > |\Im(2n\pi\theta)|,$$

t being still positive. The limitation imposed on θ is of no great consequence, because the integrals on the left of (1) and (4) are periodic in θ with period 2. The integrals on the left of (1) and (4) can also be evaluated in the extreme cases when $\Re(\theta) = \pm 1$, provided that they are interpreted as principal values and appropriate modifications are made in the series of residues on the right.

Thus, when $\Re(\theta) = +1$, half the term of the series

$$\sum \exp\{-t(2m+1+\theta)+2n\pi i(2m+1+\theta)^2\}$$

for which $m = -1$ has to be inserted, and half the term of the series

$$\sum \exp\{-t(2m+1-\theta)+2n\pi i(2m+1-\theta)^2\}$$

for which $m = 0$ has to be omitted.

The extension to complex values of t is slightly more troublesome, since

$$\int_{-\infty}^{\infty} \frac{1-e^{\pi i\theta-\pi x}}{1+e^{\pi i\theta-\pi x}} \exp(-n\pi ix^2+itx)\,dx$$

does not converge at both limits unless t is real.

However, if both θ and t are complex, with

$$|\Re(\theta)| < 1, \qquad \Re(t) > |\Im(2n\pi\theta)|, \qquad -\pi < \Im(t) \leqslant 0,$$

we can choose c so that

$$-\Im(\theta) < c < \frac{\Re(t)}{2n\pi}, \qquad +\Im(\theta) < c < \frac{\Re(t)}{2n\pi};$$

and then we consider

$$\sum_{m=1}^{\infty}(-1)^m\epsilon_m e^{\pm m\pi i\theta}\int_{c+\infty i}^{\infty}\exp\{-n\pi iz^2-(m\pi-it)z\}\,dz,$$

which we prove to be equal to

$$\int_{c+\infty i}^{\infty}\frac{-2e^{\pm\pi i\theta-\pi z}}{1+e^{\pm\pi i\theta-\pi z}}\exp(-n\pi iz^2+itz)\,dz.$$

The last integrand has the same poles and the same residues thereat as

$$\frac{1-e^{\pm\pi i\theta-\pi z}}{1+e^{\pm\pi i\theta-\pi z}}\exp(-n\pi iz^2+itz),$$

and hence, in place of (5), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}&\frac{-2e^{\pm\pi i\theta-\pi z}}{1+e^{\pm\pi i\theta-\pi z}}\exp(-n\pi iz^2+itz)\,dz\\&=4i\sum_{m=0}^{\infty}\exp\{-t(2m+1\pm\theta)+n\pi i(2m+1\pm\theta)^2\}-\\&-\frac{2}{\sqrt{n}}\exp\left(\frac{it^2}{4n\pi}-\frac{i\pi}{4}\right)\sum_{m=1}^{\infty}(-1)^{m-1}e^{\pm m\pi i\theta}\exp\left(-\frac{2mt+i\pi m^2}{4n}\right).\end{aligned}\tag{6}$$

By subtracting the two results contained in this formula, we obtain (1) in the extended domain now under consideration; and by adding them we obtain

$$\begin{aligned}\int_0^{\infty}e^{-i\pi nx^2}&\left\{\frac{\sin tx\sinh\pi x}{\cosh\pi x+\cos\pi\theta}+i\cos tx\right\}dx\\&=\sum_{m=0}^{\infty}\{e^{-(2m+1-\theta)t+(2m+1-\theta)^2i\pi n}+e^{-(2m+1+\theta)t+(2m+1+\theta)^2i\pi n}\}+\\&+\frac{1}{\sqrt{n}}\exp\left(\frac{it^2}{4n\pi}+\frac{i\pi}{4}\right)\sum_{m=1}^{\infty}(-1)^{m-1}\cos m\pi\theta\,e^{-(2mt+m^2i\pi)/(4n)},\end{aligned}\tag{7}$$

in place of (4).

It may be remarked that (1) is valid in the still more extensive domain

$$|\Re(\theta)|<1,\qquad \Re(t)>|\Im(2n\pi\theta)|,\qquad -\pi<\Im(t)<\pi.$$

The extension to positive values of $\Im(t)$ may be effected either by the theory of analytic continuation or by making certain changes in the integrands and contours which have been used hitherto.

6. The integrals studied by Mordell which were mentioned at the beginning of this paper are of the form

$$\int_{-\infty}^{\infty} \frac{e^{-Az^2+Bz}}{e^{Cz}+D}\,dz,$$

the path of integration being the real axis; in these integrals the condition $\Re(A) \geqslant 0$ is necessary for convergence, and the condition $\Re(A) > 0$ is sufficient for convergence.

If we assign to $\arg A$ its principal value, so that

$$-\tfrac{1}{2}\pi \leqslant \arg A \leqslant \tfrac{1}{2}\pi,$$

then $\Re(-Az^2) \to +\infty$ as $|z| \to \infty$ when z lies in the two quadrants defined by the inequalities

$$\tfrac{1}{4}\pi - \tfrac{1}{2}\arg A < \arg z < \tfrac{3}{4}\pi - \tfrac{1}{2}\arg A,$$
$$\tfrac{5}{4}\pi - \tfrac{1}{2}\arg A < \arg z < \tfrac{7}{4}\pi - \tfrac{1}{2}\arg A.$$

Poles of the integrand whose modulus is sufficiently large lie inside these quadrants if

$$\Re\left(\frac{A}{C^2}\right) > 0,$$

and they lie outside these quadrants if

$$\Re\left(\frac{A}{C^2}\right) < 0.$$

When the last inequality is satisfied, the path of integration can be deformed (without passing over any poles of the integrand) into two portions, both starting and ending at infinity like the contour used in § 4 of this paper; one portion starts and ends either in the quadrant in which

$$-\tfrac{1}{4}\pi - \tfrac{1}{2}\arg A < \arg z < \tfrac{1}{4}\pi - \tfrac{1}{2}\arg A,$$

or else in the quadrant in which

$$\tfrac{3}{4}\pi - \tfrac{1}{2}\arg A < \arg z < \tfrac{5}{4}\pi - \tfrac{1}{2}\arg A,$$

and the integral along this portion of the path can be transformed into an infinite series by the calculus of residues; the other portion can be drawn so that it lies wholly on one side of the straight line on which the poles of the integrand lie, and the integral along this portion of the path can be transformed into an infinite series by expanding the integrand into a series of either ascending or descending powers of e^{Cz}/D (according as $|e^{Cz}/D|$ is less than or greater than

unity throughout this portion of the contour) and integrating term by term.

When $\Re(A/C^2) > 0$, it is not possible to deform the path of integration in the manner just described, and the simplest way of evaluating the integral is Mordell's method of transforming it into one of a system of integrals discussed by Lerch.*

The integrals discussed by Ramanujan are substantially the borderline cases of the two types of integrals just described in which $\Re(A) = 0$, $\Re(A/C^2) = 0$.

A systematic discussion of the integral

$$\int_{-\infty}^{\infty} \frac{e^{-Az^2+Bz}}{e^{Cz}+D}\,dz$$

by this method of contour integration involves the investigation of rather a large number of special cases, and I propose to reserve it for another paper.

* *Rozpravy České Akademie Cisaře Františka Josefa*, 1 (1892), No. 24. I must express my indebtedness to Professor G. H. Hardy for the loan of a copy of this somewhat inaccessible paper; it is fortunate that an excellent summary of the paper, compiled by Lerch himself, is to be found in the *Jahrbuch*, 1892, 442–5.

THE DISCONTINUITIES OF AN ARBITRARY FUNCTION OF TWO VARIABLES

By U. S. HASLAM-JONES (*Liverpool*)

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1. One of the most familiar types of discontinuity of a function is that in which the limits obtained as we approach the point of discontinuity in one direction are not identical with those obtained by another path of approach. With two variables we may consider such simple examples as

$$(1) \qquad f(x,y) = \frac{x+y}{x-y};$$

$$(2) \quad \begin{cases} f(x,y) = \dfrac{1}{n} & \left(\dfrac{1}{n+1} < x < \dfrac{1}{n}\right), \\ f(x,y) = 2 & \left(x = \dfrac{1}{n}\right). \end{cases}$$

In example (1) there is a distinct limit for each direction of approach to the point $(0, 0)$; whilst in example (2) there are three distinct limits at any point (n^{-1}, y) according as we approach from the left, from the right, or along the line $x = n^{-1}$.

It was shown by W. H. and G. C. Young* that such points are comparatively rare, and that all the limits of $f(x, y)$ at a point P may be obtained by means of sequences of points tangential at P to a given half-ray τ, unless P belongs to a certain set, say D (independent of τ) which lies on a countable set of non-oscillatory† curves. This is the strict analogue of the classic theorem for functions of one variable: 'Except in a countable set of points, all the limits of $f(x)$ at any point may be obtained by approach either from the left or from the right.'

More recently, Kolmogorov and Verčenko‡ have stated a deeper theorem concerning the nature of the discontinuity at points of the

* W. H. and G. C. Young, *Proc. London Math. Soc.* (2) 17 (1918), 1–16. See also A. Kolmogorov and J. Verčenko, *Comptes rendus Ac. Sc. U.R.S.S.* 1 (1934), 106–7.

† A curve is *non-oscillatory* if it is monotone with respect to some pair of axes in the plane. All such curves are, of course, rectifiable.

‡ A. Kolmogorov and J. Verčenko, *Comptes rendus Ac. Sc. U.R.S.S.* 4 (1934), 362–4.

exceptional set D. Their theorem, a proof of which is given in Theorem 3 below, shows that, except in a set of linear measure zero, the discontinuity resembles example (2) rather than example (1). The proofs given here depend upon results obtained in an earlier paper* concerning the tangential properties of sets of points in two dimensions. The method of Kolmogorov and Verčenko appears to be different in that they derive the sets-of-points theorems as special cases of the discontinuity theorems.

The extension to the set of all limits at a point is made in Theorem 5. The complete result may be stated briefly as follows: If a function $f(z)$ is defined in an open two-dimensional region E, then at every point z of E, with the exception of a set of length zero, either (i) the limits are the same for every direction of approach to z; or (ii) there is a line through z such that the limits obtained by approaching z on one side of this line are the same for every direction, and the limits on the other side are the same for every direction.

2. Let $f(z) = f(x+iy)$ be a function of the two real variables x and y. In order to avoid difficulties at points in whose neighbourhood the function is undefined, we shall suppose that $f(z)$ is defined in an open set E.

Let $S_\theta^\phi(z, \rho)$ denote the open sector of points w satisfying the inequalities

$$0 < |w-z| < \rho, \qquad \theta < \operatorname{am}(w-z) < \phi.$$

We write

$$B_\theta^\phi(z, \rho) = \overline{\operatorname{bound}}\, f(w), \qquad w \in S_\theta^\phi(z, \rho),$$

$$M_\theta^\phi(z) = \lim_{\rho, \eta \to +0} B_{\theta-\eta}^{\phi+\eta}(z, \rho),$$

$$M_\alpha(z) = M_\alpha^\alpha(z),$$

$$M(z) = \overline{\lim_{w \to z}}\, f(w) = M_0^{2\pi}(z).$$

(The order in which the limits are taken in M_θ^ϕ is immaterial.)

It is easily seen that

$$\overline{\operatorname*{bound}_{\theta \leqslant \alpha \leqslant \phi}}\, M_\alpha = M_\theta^\phi;$$

also that

$$M_\alpha \geqslant \overline{\lim_{\beta \to \alpha}}\, M_\beta,$$

hence M_α is an upper semi-continuous function of α and attains its

* See above, pp. 116–23. This paper will be referred to as 'T.P.'

upper bound in any closed interval. It follows that there is some value of α satisfying

$$M_\alpha = M_\theta^\phi \quad (\theta \leqslant \alpha \leqslant \phi).$$

Similarly we may define lower limits $m(z)$, $m_\theta^\phi(z)$, $m_\alpha(z)$.

3. The following theorem is due to W. H. Young:* the proof, which is short, is included for the sake of completeness.

THEOREM 1. *The set of points at which $f(z) > M(z)$ is countable.*

If $f(z) > M(z)$, there is a rational number r such that

$$f(z) > r > M(z).$$

Let r_1, r_2,... be the rational numbers arranged in sequence, and let E_n be the set in which $f(z) > r_n > M(z)$. All points of E_n are isolated points; hence E_n is a countable set. The set in which $f(z) > M(z)$ is identical with $\sum E_n$, and is therefore countable also.

In the succeeding theorems certain of the ideas and results of T.P. are employed. For reference these are summarized below: *Notation.* A linearly measurable set lying on a countable infinity of non-oscillatory curves is called an L_σ: a set of linear measure zero is called an l_0. A plane set of points E is said to have a *void sector* (α, β) at z if (i), given any positive η, there is a positive ρ such that the open sector $S_{\alpha+\eta}^{\beta-\eta}(z, \rho)$ contains no point of E, and (ii) (α, β) is not a part of a larger void sector. Those limit points of E at which there is a void sector are called *boundary points* of E.

THEOREM A. (i) The boundary points of E form an L_σ;

(ii) the points at which there is a void sector of magnitude greater than π form a countable set;

(iii) the points at which there is a void sector of magnitude less than π form an l_0.

That is to say, at all boundary points, with the exception of points of an l_0, there are either one or two void sectors each of magnitude π.

THEOREM 2. (W. H. and G. C. Young.) *For all values of α, $M_\alpha(z) = M(z)$ except at points of an L_σ, independent of α.*

With the notation of Theorem 1, let E_n denote the set in which $f(z) > r_n$, let $\mathcal{E}_n$ denote the set of boundary points of E_n; except at points of $\sum \mathcal{E}_n$, $M_\alpha(z) = M(z)$ for every value of α. By T. P. (Theorem A), the set $\mathcal{E}_n$ is an L_σ: hence $\sum \mathcal{E}_n$ is also an L_σ, and the theorem is proved.

* W. H. Young, *Proc. London Math. Soc.* (2) 8 (1910), 117–24 (Theorem 1).

If we combine this theorem with a corresponding m-theorem, we see that, with the exception of the points of an L_σ, the upper and lower limits at any point can be obtained by approach in any direction. The next theorem goes further and shows that, except at points of an l_0, there are at most three values for M_α.

THEOREM 3. (Kolmogorov and Verčenko.) *Except at points of an l_0, there is a direction $\omega = \omega(z)$ and numbers $M' = M'(z)$, $M'' = M''(z)$, such that*

$$M_\alpha = M' \quad (\omega < \alpha < \omega+\pi),$$
$$M_\alpha = M'' \quad (\omega-\pi < \alpha < \omega),$$
$$M_\omega = M_{\omega+\pi} = M,$$

where $M'' \leqslant M' \leqslant M$.

Consider again the set $\mathcal{E}_n$, defined in Theorem 2. It follows from T.P. (Theorem A) that at all points of a set $\mathcal{E}_n'$ which differs from $\mathcal{E}_n$ at most by an l_0 there are either one or two sectors of magnitude π void of E_n. If there is only one such void sector at z, then there is a direction ω_n such that every sector $S_\theta^\phi(z, \rho)$ for which

$$\omega_n-\pi \leqslant \theta < \phi \leqslant \omega_n$$

contains points of E_n, whilst, if $\omega_n < \theta < \phi < \omega_n+\pi$, there is some number ρ for which $S_\theta^\phi(z, \rho)$ is void of E_n. This implies the inequalities

$$M_\alpha \geqslant r_n \qquad (\omega_n-\pi \leqslant \alpha \leqslant \omega_n),$$
$$M_\alpha \leqslant r_n \qquad (\omega_n < \alpha < \omega_n+\pi).$$

If there are two void sectors at z, there is a direction ω_n such that, if $\theta < \omega_n < \phi$ or $\theta < \omega_n+\pi < \phi$, every sector $S_\theta^\phi(z, \rho)$ contains points of E_n, whilst, if $\omega_n-\pi < \theta < \phi < \omega_n$ or $\omega_n < \theta < \phi < \omega_n+\pi$, there is a number ρ for which $S_\theta^\phi(z, \rho)$ is void of E_n. In this case

$$M_\alpha \leqslant r_n \qquad (0 < |\alpha-\omega_n| < \pi),$$
$$M_\alpha \geqslant r_n \qquad (\alpha = \omega_n \quad \text{or} \quad \omega_n+\pi).$$

The direction ω_n is independent of n: for, if z belongs to $\mathcal{E}_m$ and to $\mathcal{E}_n$, where $r_m > r_n$, and $\omega_m \neq \omega_n$, it is easily seen that there will be a direction α in which $M_\alpha \leqslant r_n$ and $M_\alpha \geqslant r_m$, which is impossible. We may suppress the suffix and write $\omega_n = \omega$.

Let z be a point of $\mathcal{E}' = \sum \mathcal{E}_n'$, which differs from $\sum \mathcal{E}_n$ only by an l_0. Let $M(z)$ and $M'(z)$ be the upper and lower bounds of r_n for which E_n has a limit point at z with two void sectors. Then $M'(z)$ is also the upper bound of r_n for which E_n has one void sector only: let $M''(z)$ be the lower bound. The equalities stated in the theorem clearly

hold, and the theorem is thus proved for points of $\sum \mathcal{E}_n$. At all other points Theorem 2 shows that the equalities are still true, and $M'' = M' = M$.

We shall call the line through z in the direction ω the *critical line* at z. It is convenient sometimes to speak of the critical line even when $M'' = M' = M$: in this case its direction is arbitrary, and may be regarded as coincident with any given direction.

THEOREM 4. *If Γ is any non-oscillatory curve, then at every point of Γ, with the exception of an l_0, the critical line of $f(z)$ coincides with the tangent to Γ.*

Suppose that the theorem is false: then we can find a finite stretch on Γ, from z_0 to z_1, say γ, in which the tangent and the critical line differ on a set of positive length. If z_0 and z_1 are joined by two regular curves, one on either side of γ, then these two curves enclose a region R which is divided by γ into two regions R_1 and R_2. Consider a function $\phi(z)$ which is equal to $f(z)$ in R_1 and on γ, and to $f(z)-1$ in R_2. Let ζ be a point of Γ at which there is a tangent and at which the critical direction for $f(z)$ differs from the tangent. Since the curve has a tangent almost everywhere, the set of such points is of positive measure. But, at ζ, $M_\alpha \phi$ has the values $M''f$, $M''f-1$, $M'f$ for distinct ranges of values of α. Since, by Theorem 3, this is not possible except on a set of zero length there is a contradiction, and the theorem is established.

COROLLARY. *Except at points of an l_0, the critical line at z coincides with the tangent to any one of the non-oscillatory curves through z on which the exceptional set $\mathcal{E}'$ lies.*

This corollary appears to be less general than the theorem, but actually this is not so. It can be shown that the common points of two non-oscillatory curves at which the tangents do not coincide form an l_0. The points of any such curve at which there is a definite critical direction lie on $\mathcal{E}'$, and except on an l_0, the tangent to the curve must coincide with the tangent to $\mathcal{E}'$.

The theorems for lower limits corresponding to Theorems 1–4 are proved in a similar manner.

4. The set of limits at a point

Let $\mathcal{L}_\alpha(z)$ denote the set of all limits of $f(z)$ which can be obtained by approach to the point z in the direction α. If we denote by

$F = F(z, \alpha-\eta, \alpha+\eta, \rho)$ the set of values of $f(z)$ in $S_{\alpha-\eta}^{\alpha+\eta}(z, \rho)$, together with the limit points of the set, then

$$\mathscr{L}_\alpha(z) = \lim_{\eta,\rho\to 0} F(z, \alpha-\eta, \alpha+\eta, \rho).$$

Since F is a closed set and $\mathscr{L}_\alpha$ is the limit of a decreasing sequence of such sets, it follows that $\mathscr{L}_\alpha$ is also a closed set. Let $\mathscr{L}(z)$ denote the set of all limits at the point z.

THEOREM 5. (i) *Except at points of an L_σ, $\mathscr{L}_\alpha = \mathscr{L}$ for all values of α;*

(ii) *except at points of an l_0, there is a direction ω and three sets of limits $\mathscr{L}$, $\mathscr{L}'$, $\mathscr{L}''$ such that*

$$\mathscr{L}_\alpha = \mathscr{L}' \quad (\omega < \alpha < \omega+\pi),$$

$$\mathscr{L}_\alpha = \mathscr{L}'' \quad (\omega-\pi < \alpha < \omega),$$

$$\mathscr{L}_\omega = \mathscr{L}_{\omega+\pi} = \mathscr{L}, \quad \text{and} \quad \mathscr{L}' \prec \mathscr{L}, \quad \mathscr{L}'' \prec \mathscr{L}.$$

Let ζ be a point at which there is a limit ν in $\mathscr{L}$ which does not belong to $\mathscr{L}_\alpha$ for some value of α, say $\alpha = \beta$. We show that ζ belongs to a certain L_σ independent of β and ν, and that, except on an l_0, there is a direction ω independent of β and ν, such that ν belongs to $\mathscr{L}_\alpha$ for $\alpha = \omega$ and for $\alpha = \omega+\pi$, and ν does not belong to $\mathscr{L}_\alpha$ for $\omega < \alpha < \omega+\pi$. The theorem is an immediate consequence of this result.

Either $m_\beta < \nu$ or $M_\beta > \nu$: suppose $m_\beta < \nu$. Since $\mathscr{L}_\beta$ is a closed set, there are numbers p, q such that (i) $p < \nu < q$, (ii) p belongs to $\mathscr{L}_\beta$, (iii) no other number between p and q belongs to $\mathscr{L}_\beta$. Let r and r' be rational numbers such that $r' < r$, and let the function $f_{r,r'}(z)$ be defined by

$$f_{r,r'} = f \quad (f < r),$$

$$f_{r,r'} = r' \quad (f \geqslant r).$$

If r and r' are chosen so that $r' < p < \nu < r < q$, then

$$M_\beta f_{r,r'}(\zeta) = p < \nu, \qquad \nu \leqslant M f_{r,r'}(\zeta).$$

Hence, by Theorems 2 and 3, ζ belongs to an L_σ, say $\Gamma_{r,r'}$: also, unless ζ belongs to a certain l_0, say $\gamma_{r,r'}$, there is a direction $\omega_{r,r'}$ such that

$$M_\alpha f_{r,r'}(\zeta) = p \qquad (\omega_{r,r'} < \alpha < \omega_{r,r'}+\pi),$$

$$\nu \leqslant M_\alpha f_{r,r'}(\zeta) \leqslant r \qquad (\alpha = \omega_{r,r'} \quad \text{or} \quad \omega_{r,r'}+\pi).$$

Hence, except in $\gamma_{r,r'}$, ν is not a member of $\mathcal{L}_\alpha f$ for $\omega_{r,r'} < \alpha < \omega_{r,r'}+\pi$ but there is a member of $\mathcal{L}_\alpha f$ between ν and r for $\alpha = \omega_{r,r'}$ or $\alpha = \omega_{r,r'}+\pi$.

The sums $\sum \Gamma_{r,r'}$ and $\sum \gamma_{r,r'}$ taken over all rational values of r and r' form an L_σ, say Γ, and an l_0, say γ, respectively. Also $\omega_{r,r'}$ is independent of r and r', except possibly on an l_0, for, if not, there are rationals r, r', s, s' such that $\omega_{r,r'} \neq \omega_{s,s'}$ on a set of positive length. This set is contained in the set Γ and must therefore be of positive length on one at least of the countably infinite set of non-oscillatory curves which contain Γ. By Theorem 4, the directions $\omega_{r,r'}$ and $\omega_{s,s'}$ are tangential to any such curve, and therefore coincident, except on an l_0. Hence there is a contradiction, and, except on an l_0, say γ', we may write $\omega_{r,r'} = \omega$.

Since there are values of r arbitrarily near to ν, and for all such values, except in the set $\gamma+\gamma'$, there are members of the closed sets $\mathcal{L}_\omega f$ and $\mathcal{L}_{\omega+\pi} f$ between ν and r, it follows that ν belongs to $\mathcal{L}_\omega f$ and to $\mathcal{L}_{\omega+\pi} f$. We have shown that ν does not belong to $\mathcal{L}_\alpha$ for $\omega < \alpha < \omega+\pi$.

If $M_\beta > \nu$, a corresponding proof can be given. The results stated in the theorem are thus proved.

SOME COMBINATORIAL PROPERTIES OF MEASURABLE SETS

By J. GILLIS (*Sunderland*)

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1. Elsewhere* I have proved the following results.

THEOREM I. *Given an infinite sequence* $\{A_n\}$ *of measurable sets all contained in the interval* $(0, 1)$ *and such that, for each* n,

$$mA_n \geqslant \alpha \quad (0 < \alpha < 1), \tag{1.1}$$

then, for each positive ϵ *and each positive integer* k, *we can find* k *distinct positive integers*† $n_1, \ldots, n_{k-1}, n_k$, *such that*

$$m(A_{n_1} \times A_{n_2} \times A_{n_3} \times \ldots \times A_{n_k}) > \alpha^k - \epsilon. \tag{1.2}$$

THEOREM II. *Given a real number* α *such that* $0 < \alpha < 1$ *we can find an infinite sequence* $\{A_n\}$ *of measurable sets contained in* $(0, 1)$, *satisfying* (1.1), *and such that, for each set of* k *distinct natural numbers* $n_1, n_2, \ldots, n_k$,

$$m(A_{n_1} \times A_{n_2} \times \ldots \times A_{n_k}) < \alpha^k.$$

These theorems suggest the problem of what happens if the family of sets is non-denumerable. I prove the following theorem.

THEOREM III. *Suppose that we have a class* $\mathfrak{F}$ *of measurable linear sets all contained in the interval* $(0, 1)$ *and such that*

(i) $\mathfrak{F}$ *has cardinal* $\mathfrak{c}$,

(ii) *for each set* E *of* $\mathfrak{F}$, $mE \geqslant \alpha \quad (0 < \alpha < 1)$,

then, given a positive ϵ, *we can find*

(iii) *a subclass* $\mathfrak{H}$ *of* $\mathfrak{F}$, *also of cardinal* $\mathfrak{c}$, *and such that the common part of any two members of* $\mathfrak{H}$ *has measure greater than* $\alpha - \epsilon$, *and*

(iv) *a denumerably infinite subclass* $\mathfrak{K}$ *of* $\mathfrak{H}$ *such that the set of points common to all its members has measure greater than* $\alpha - \epsilon$.

I show in § 3, assuming the 'hypothesis of the continuum', that this result is the best possible. In §§ 4, 5 I discuss a new set of ideas, construct an example which seems to be of interest, and continue in § 6 with a statement of what appears to be the most immediate

* 'Note on a property of measurable sets', *J. of London Math. Soc.* (in the press).

† Actually (see loc. cit.) the proof is given in detail only for the case $k = 2$, but the proof of the general theorem is almost identical.

problem. Finally, in § 7, I give an application of Theorem III to the theory of the Carathéodory linear measure of plane sets.

2. *Proof of Theorem III.* If A, B are two measurable sets in the interval $(0, 1)$ then I define their *relative displacement* by

$$\delta(A, B) = m(A - A \times B) + m(B - A \times B). \tag{2.1}$$

If $f_A(x)$, $f_B(x)$ are the respective characteristic functions of A, B we get

$$\delta(A, B) = \int_0^1 |f_A(x) - f_B(x)|\, dx. \tag{2.2}$$

I now denote by Σ the class of all measurable sets in $(0, 1)$, with the proviso that sets A, B for which $\delta(A, B) = 0$ are to be regarded as identical. Then it is clear from (2.1) and (2.2) that

(*a*) $\delta(A, B) = \delta(B, A) \geqslant 0$;

(*b*) $\delta(A, B) = 0$ if and only if $A = B$;

and (*c*) $\delta(A, B) + \delta(B, C) \geqslant \delta(A, C)$.

Hence Σ is a *metric space.* Moreover, we see from (2.2) that Σ is *complete.** Now consider the class of finite sets of rational intervals. This class is denumerable and it is clearly everywhere dense in Σ which is, therefore, *separable.*†

Take now the class $\mathfrak{F}$ of the theorem. It has cardinal $\mathfrak{c}$ and lies in the complete and separable space Σ. It follows that $\mathfrak{F}$ contains a *point of condensation,*‡ E (say). Let $\mathfrak{H}$ be the set of members A of $\mathfrak{F}$ such that

$$\delta(E, A) < \tfrac{1}{2}\epsilon. \tag{2.3}$$

Then $\mathfrak{H}$ has cardinal $\mathfrak{c}$. Also, if A and B are any two members of $\mathfrak{H}$, we see from (2.3) and the properties of δ that

$$\delta(A, B) < \epsilon. \tag{2.4}$$

(2.4) and (2.1) give $m(A - A \times B) < \epsilon$, whence, by hypothesis (ii),

$$m(A \times B) > \alpha - \epsilon. \tag{2.5}$$

This establishes (iii). To prove (iv) I make the remark that, for each positive integer n, we can find a member A_n of $\mathfrak{H}$ such that

$$\delta(E, A_n) < \epsilon/2^n.$$

* For the notions 'complete' and 'metric', cf. Hausdorff, *Mengenlehre* (Berlin, 1927). The proof of completeness is immediate, e.g. by the Riesz-Fischer Theorem.

† Cf. Hausdorff, op. cit., Chap. VI.

‡ Hausdorff, loc. cit.

Then, if $E-E\times A_n = B_n$ (say), we get $mB_n < \epsilon/2^n$. Let $\sum_{n=1}^{\infty} B_n = B$; then

$$mB < \epsilon. \tag{2.6}$$

But, for each n, $\quad E-B \subset E-B_n \subset A_n$,

and so

$$E-B \subset \prod_{n=1}^{\infty} A_n. \tag{2.7}$$

Since E is contained in $\mathfrak{F}$ we deduce from (2.6), (2.7), and hypothesis (ii) that

$$m\Big[\prod_{n=1}^{\infty} A_n\Big] > \alpha-\epsilon.$$

Now let $\mathfrak{R}$ denote the sequence of sets $\{A_n\}$ and the theorem is completed.

3. In this section I consider whether Theorem III is the best possible. The most important question relates to (iv), viz.:'Is it possible or not possible to find $\mathfrak{R}$ having cardinal $\mathfrak{c}$'. I shall show that, without some further restriction on the family $\mathfrak{F}$, the answer is negative. Assuming,* for simplicity, the hypothesis of the continuum, suppose that the points of the interval $(0, 1)$ are ordered thus:

$$a_1, a_2, \ldots, a_\xi, \ldots \quad (\xi < \Omega). \tag{3.1}$$

Let A_η denote the set of terms a_ξ of this series such that $\xi > \eta$. Then each A_η has measure 1, while there is no point common to more than a denumerable infinity of A_η's.

4. Suppose now that the linear segment defined by $x = x_0$, $0 \leqslant y \leqslant 1$ contains, for each x_0 in $(0, 1)$, a measurable set E_{x_0}. Then a set P in $0 \leqslant y \leqslant 1$ will be said to *correspond* to a set Q in $0 \leqslant x \leqslant 1$ if, for each x of Q, E_x contains P. I now put the following problem:

Suppose that there exists a fixed positive number α such that, for each x of $(0, 1)$, $mE_x \geqslant \alpha$. Suppose also† that the plane set $\sum_{0\leqslant x\leqslant 1} \tilde{E}_x$ is measurable-L. Does it follow that there must exist sets P, Q, both of positive exterior measure, such that P corresponds to Q?

In § 5 I shall show that the answer to this question is in the

* The assumption is not strictly necessary but the simplification involved is interesting. For some interesting properties of the *gegenbeispiel* of this section see Sierpinski, *Hypothèse du Continu* (Warsaw, 1934), pp. 23–4.

† I use the respective notations E_{x_0} and $\tilde{E}_{x_0}$ to indicate the set of points on the line $x = x_0$ regarded firstly as a set of values of y and secondly as a plane set.

negative, though the question itself is of interest. But, before proceeding to it, I shall show that the *gegenbeispiel* of § 3 cannot be made to solve this problem. For, let

$$b_1, b_2, \ldots, b_\xi, \ldots \quad (\xi < \Omega) \tag{4.1}$$

be any arrangement of the real numbers in $(0, 1)$ as an ω_1. In particular (4.1) may coincide with (3.1). Then, for each x in $(0, 1)$ there is a unique ξ such that $x = b_\xi$. Put $E_x = A_\xi$. Let $E = \sum_{0 \leqslant x \leqslant 1} \tilde{E}_x$ and let CE denote its complementary set with respect to the unit square. Then the intersection of CE with each ordinate $x =$ constant is denumerable and so is the intersection of E with each abscissa $y =$ constant. It follows that, whatever the arrangements (3.1) and (4.1), E is non-measurable.

5. To provide now a negative answer to the question propounded in § 4 I shall in fact show that the following is possible:

(1) For each x in $(0, 1)$, $mE_x = 1$.

(2) The plane set $\sum_{0 \leqslant x \leqslant 1} \tilde{E}_x$ is measurable-L.

(3) For each set Q of positive exterior measure, $\prod_{x \in Q} E_x$ has measure zero.

The construction will be best explained in stages.

5.1. *Definition of the operation $T(\alpha, \nu)$ on a rectangle.* Suppose that we have a rectangle R defined by $a \leqslant x \leqslant b$, $h \leqslant y \leqslant k$. If α is any positive number and ν a positive integer, the operation $T(\alpha, \nu)$ on R is defined as follows:

(i) divide the rectangle R into ν equal rectangles $R_1^*, R_2^*, \ldots, R_\nu^*$, where R_i^* is given by

$$a \leqslant x \leqslant b, \qquad h + \frac{i-1}{\nu}(k-h) \leqslant y \leqslant h + \frac{i}{\nu}(k-h);$$

(ii) from each R_i^* take the rectangle R_i given by

$$a \leqslant x \leqslant b, \qquad h + \frac{i-1}{\nu}(k-h) \leqslant y \leqslant h + \frac{i-1+\alpha}{\nu}(k-h).$$

The set of rectangles $R_1, R_2, \ldots, R_\nu$ will then be called $T(\alpha, \nu).R$. I shall denote by $S(\alpha, \beta)$ the strip $\alpha \leqslant x \leqslant \beta$, $0 \leqslant y \leqslant 1$.

5.2. *The sequences $\{m_k\}$, $\{n_k\}$, $\{a_k\}$.* I shall need two sequences $\{m_k\}$, $\{n_k\}$ of natural numbers and a sequence $\{a_k\}$ of positive numbers

which satisfy the following conditions:

(i) $$0 < a_k < 1 \quad (k = 1, 2, \ldots), \tag{5.1}$$

(ii) $$\prod_{k=1}^{\infty} a_k = a > 0, \tag{5.2}$$

(iii) $$\lim_{k \to \infty} a_k^{n_k} = 0, \tag{5.3}$$

(iv) if $i < j < k$, then $\dfrac{a_i n_k}{n_j}$ is an integer.

5.3. *First step of construction.* To begin with, write S_i for

$$S\left(\frac{i-1}{m_1}, \frac{i}{m_1}\right) \quad (i = 1, 2, \ldots, m_1),$$

and let $$T^{(1)} = \sum_{i=1}^{m_1} T(a_1, n_i) . S_i. \tag{5.4}$$

Again, let S_{ij} denote the strip

$$S\left(\frac{i-1}{m_1} + \frac{j-1}{m_1 m_2}, \frac{i-1}{m_1} + \frac{j}{m_1 m_2}\right) \quad (1 \leqslant i \leqslant m_1;\ 1 \leqslant j \leqslant m_1 m_2),$$

and write $$T^{(2)} = \sum_{i,j} T(a_2, n_{m_1+j}) . S_{ij}.$$

In general write $S_{i_1 i_2 \ldots i_r}$ for the strip

$$S\left(\frac{i_1-1}{m_1} + \frac{i_2-1}{m_1 m_2} + \ldots + \frac{i_r-1}{m_1 m_2 \ldots m_r}, \frac{i_1-1}{m_1} + \frac{i_2-1}{m_1 m_2} + \ldots + \frac{i_r}{m_1 m_2 \ldots m_r}\right),$$

and $$T^{(r)} = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \ldots \sum_{i_r=1}^{m_r} T(a_r, n_{m_1+m_2+\ldots+m_{r-1}+i_r}) . S_{i_1 \ldots i_r}, \tag{5.5}$$

and finally $$T = \prod_r T^{(r)}. \tag{5.6}$$

5.4. I now assert

(1′) T is a closed set.

(2′) T has plane Lebesgue measure a.

(3′) If Q is a set of positive exterior measure contained in the linear interval $0 \leqslant x \leqslant 1$, then the set of values of y such that $(x, y) \in T$ for all x in Q has measure zero.

Since each $T^{(r)}$ is the sum of a finite set of closed rectangles and therefore itself closed, (1′) is an immediate consequence of (5.6).

Now, if α is positive and ν a natural number, I denote by $M_{\alpha,\nu}$ the linear set in $(0, 1)$ consisting of ν equal and equally distributed intervals each of length α/ν such that $M_{\alpha,\nu}$ contains the origin. Let

x_0 be any number in $(0, 1)$ not of the form $\sum_{i=1}^{N} \frac{\lambda_i}{m_1 \ldots m_i}$ with finite N and positive integral λ_i. Then it is clear from (5.5) and (5.6) that the intersection $T_{x_0}^{(r)}$ of each $T^{(r)}$ with the ordinate $x = x_0$ is of the form M_{α_r, ν_r} where $\nu_r = \nu_r(x_0)$. In view of (iv) of 5.3 we easily deduce

$$m\Big[\prod_{r=1}^{q} T_{x_0}^{(r)}\Big] = a_1 a_2 \ldots a_q, \tag{5.7}$$

for every positive integer q. In particular the intersection of T with $x = x_0$ has measure $\prod_{r=1}^{\infty} a_r = a$. Since, by (1′), T is measurable, (2′) follows by Fubini's Theorem.

Proof of (3′). Since Q has positive exterior measure we can find a sequence $i_1, i_2, \ldots, i_r$, of positive integers such that, for each r, the mean density of Q in the interval $S_{i_1 i_2 \ldots i_r}$ is greater than $\frac{1}{2}$. It follows that, of the m_{r+1} sub-intervals of $S_{i_1 \ldots i_r}$ which are used towards the construction of $T^{(r+1)}$, at least $\frac{1}{2} m_{r+1}$ contain points of Q. We deduce, by an argument similar to that which established (5.7), that

$$m\Big[\prod_{x \in Q} T_x^{(r+1)}\Big] \leqslant a_{r+1}^{\frac{1}{2} m_{r+1}}. \tag{5.8}$$

Since (5.8) is true for every positive integer r it may be combined with (5.3) and (5.6) to give (3′).

5.5. By (1′) we can write the set complementary to T with respect to the unit square as $\sum_{i=1}^{\infty} U_i$ where each U_i is a square with sides parallel to the axes of coordinates. We place in U_i a set T_i^* similar to T and whose relation to U_i is similar to that of T to the unit square. Finally, let $T_1 = \sum_{i=1}^{\infty} T_i^*$. We see that

$$\begin{aligned} m(T+T_1) &= mT + \sum_{i=1}^{\infty} mT_i^* \\ &= a(1 + \sum_i mU_i), \text{ by (1′)}, \\ &= a(2-a) \\ &= 1-(1-a)^2. \end{aligned} \tag{5.9}$$

Similarly, we define T_2 from the squares complementary to $T+T_1$ and we see that

$$m(T+T_1+T_2) = 1-(1-a)^3. \tag{5.10}$$

We can proceed in this way indefinitely, defining $T_3,\ldots, T_r,\ldots$, and get

$$m(T+\sum_{r=1}^{n} T_r) = 1-(1-a)^n, \tag{5.11}$$

and finally, by (5.2), $$m(T+\sum_{r=1}^{\infty} T_r) = 1. \tag{5.12}$$

Now, since each T_r clearly possesses the property (3′) of T, *so also** *does the set* $T+\sum_{r=1}^{\infty} T_r$. By (5.12) and Fubini's Theorem we see that the intersection of this set with $x = x_0$ has measure 1 for every x_0 of $(0, 1)$ except possibly for a set X of values of x_0, where

$$mX = 0. \tag{5.13}$$

Now, if x_0 does not belong to X, let E_{x_0} be the intersection of $x = x_0$ with $T+\sum_{r=1}^{\infty} T_r$. If x_0 belongs to X, let E_{x_0} be the whole segment $x = x_0$, $0 \leqslant y \leqslant 1$. What we have proved about $T+\sum_{r=1}^{\infty} T_r$ combined with (5.13) shows that the sets have the properties (1), (2), and (3) of § 5.

6. In view of the negative answer to the problem put in § 4 the following problem arises more or less naturally. Suppose that E is a measurable plane set of Lebesgue measure 1 contained in the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. *Does there necessarily exist a perfect set Q in the interval $0 \leqslant x \leqslant 1$ such that the set of y with the property that $(x, y) \in E$ for every x of Q has positive measure?* It is easy to verify that such a set exists in the example constructed in § 5.

7. In this section I apply Theorem III to the theory of the linear measure of plane sets of points. It is clear from the proof of that theorem that it will hold, if the point-sets, instead of being contained in a linear segment, are sub-sets of a general linearly measur-

* This last step actually requires proof. It is easily seen that, for each n, $T+\sum_{r=1}^{n} T_r$ has the property (3′), and the passage to the limit follows obvious lines. I shall not go here into the details of the proof as they can easily be supplied by the reader. Note, however, that $\prod_{x\in Q}\left[T+\sum_{r=1}^{\infty} T_r\right]^{(x)}$ is certainly not, in general, identical with $\prod_{x\in Q} T^{(x)}+\sum_{r=1}^{\infty}\prod_{x\in Q} T_r^{(x)}$. See also Lomnicki and Ulam, *Fund. Math.* 22 (1934), 237–78; (247–9).

able plane set. All the properties of the function $\delta(A, B)$ hold, while the space analogous to Σ is still separable and complete, the everywhere-dense set now being the parts of the original plane set contained in finite sets of rational circles. Now the following theorem is known:

THEOREM.* *If A is a linearly measurable plane set of upper density $\frac{1}{2}$ and positive lower density and α is any number less than $\frac{1}{4}$, then there exists a set of directions θ of measure not less than $\frac{1}{2}\pi(1-4\alpha)/(1-\alpha)$ on each θ of which a sub-set A_θ of A of linear measure exceeding αLA has projection of zero Lebesgue measure.* Combining this with Theorem III we see that we can find an infinite set $\theta_1, \theta_2, \ldots, \theta_n, \ldots$ of directions such that $\prod_{n=1}^{\infty} A_{\theta_n}$ has positive linear measure. This set will then have zero projection on these directions. We deduce the following:

THEOREM. *If A is a linearly measurable plane set of upper density $\frac{1}{2}$ and positive lower density, then we can write $A = \sum_{n=1}^{\infty} A_n$ where each A_n has zero projection on an infinity of directions.*

* Cf. J. Gillis 'On linearly measurable plane sets', *C. R. de la Soc. des Sci. et des Lettres de Varsovie* (in the press).

ELECTRICAL NOTES

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VI. ENERGY DISTRIBUTION OF ELECTRONS IN A GAS

When electrons move through a gas in a uniform electric field, they acquire a kinetic energy greater than that of the molecules of the gas, which was first put in evidence by Townsend.* I found the distribution of energy amongst the electrons theoretically† on the assumption that k, the ratio of the mean energy of electrons and molecules, does not differ much from 1. The restriction has since been shown to be unnecessary. I had been under the impression not only that my formula would fail when k is large, but also that the distribution would contract indefinitely. Townsend‡ showed that there is a distribution of energy with finite spread in strong fields, and about the same time an exact formula for the distribution was given by Druyvesteyn.§ My formula has since been rediscovered by Davydov,‖ who also adapted his proof so as to give Druyvesteyn's formula. I now round off the matter by showing directly that Druyvesteyn's formula is a special case of mine.

The equations used are in § 11 of my paper. Let $F\,du$ be the number of electrons with a resultant velocity between u and $u+du$, when the small second term of equation (55) is neglected. Then F is a multiple of $p^2(p^2+b)^b\exp(-p^2)$, where b is a constant and $p^2 = hmu^2$. If b is large, the function has already gone over its maximum and become small by the time that $p^2 = b$. Hence

$$\log(F/p^2) = \text{const.}+b\log(1+p^2/b)-p^2 = \text{const.}-p^4/2b$$

approximately, so that $F \propto p^2\exp(-p^4/2b)$, where

$$b = 2e^2h^2E^2M/3N^2\pi^2\sigma^4m.$$

This is Druyvesteyn's formula in a different notation.

Returning to the general formula, the fraction of the total number of electrons whose resultant velocity lies between fractions x and

* J. S. Townsend, *Proc. Roy. Soc.* A, 81 (1908), 464.
† F. B. Pidduck, *Proc. London Math. Soc.* 15 (1915), 89.
‡ J. S. Townsend, *Phil. Mag.* 9 (1930), 1145.
§ M. J. Druyvesteyn, *Physica*, 10 (1930), 61; 1 (1934), 1003.
‖ P. Davydov, *Phys. Zeits. der Sowjetunion*, 8 (1935), 59.

$x+dx$ of the root-mean-square velocity is $F\,dx$, where

$$F = (\tfrac{3}{2}k)^{\frac{1}{2}}p^2(p^2+b)^b\exp(-p^2) \div \int_0^\infty p^2(p^2+b)^b\exp(-p^2)\,dp, \quad (1)$$

$$p^2 = \tfrac{3}{2}kx^2. \quad (2)$$

The values of F when $b = 1, 2, 3$ are shown in the following table, to which are added Maxwell's distribution ($b = 0$) and Druyvesteyn's distribution.

F

x	$k = 1$	$k = 1{\cdot}40$	$k = 1{\cdot}65$	$k = 1{\cdot}86$	k *large*
0·2	0·156	0·109	0·102	0·099	0·083
0·4	0·522	0·420	0·397	0·385	0·327
0·6	0·870	0·815	0·791	0·778	0·696
0·8	0·983	1·074	1·080	1·081	1·062
0·816	1·017*	..	..	..	..
0·878	..	1·099*	..	..	..
0·898	..	..	1·120*	..	..
0·909	..	..	..	1·131*	..
0·978	..	..	..	..	1·204*
1·0	0·925	1·067	1·077	1·096	1·201
1·2	0·689	0·774	0·805	0·824	0·962
1·4	0·430	0·449	0·457	0·462	0·498
1·6	0·228	0·207	0·200	0·195	0·147
1·8	0·104	0·077	0·067	0·062	0·038
2·0	0·042	0·023	0·018	0·015	0·001
2·5	0·002	0·000	0·000	0·000	0·000

* Maximum.

The ratio of $k-1$ to v^2/Ω^2, where v is the mean velocity of drift and Ω the root-mean-square velocity of the gas molecules, tends to $3/2\sqrt{2}$ as b tends to infinity. It varies, therefore, only between 1·18 and 1·06 for elastic collisions.

A theory of the same accuracy as that of elastic collisions was given in my paper for smooth spheres of any coefficient of restitution. The distribution is that corresponding to the value of k which would be acquired in the same electric field with perfect restitution. Thus if we wish (apart from the defects of the assumed kinetic model) to find the distribution in an experiment of Townsend with hydrogen, in which k was found to be 3·1, we must work as if k was 12·2, the value calculated for elastic collisions. The reduction of k in diatomic gases is large, but the fraction of the energy of an electron lost in

one collision is nevertheless quite small in weak electric fields. It begins at about 1 part in 400. The losses in inert gases are still smaller, and in Townsend's experiments on helium k begins by corresponding nearly to elastic collisions.

VII. SPACE CHARGES IN A MAGNETIC FIELD

1. Motion of an outward stream of electrons between a filament and a coaxial cylindrical sheath

We neglect the magnetic field of the electrons and take them to move in a radial electric field $-\partial V/\partial r$ and a uniform magnetic field H parallel to the filament, where the scalar electromagnetic potential V is in general a function of r and t. Let u be the radial velocity, v the transverse velocity, n/r the number of electrons per cm.^{3}, $-e$ the charge and m the mass of each. Let

$$\omega = He/mc \tag{1}$$

and let the radius of the filament be a, the electrons leaving it with no velocity. The equation of transverse motion being

$$\frac{\partial v}{\partial t}+\frac{u\,\partial v}{\partial r}+\frac{uv}{r}=\omega u, \quad \text{i.e.} \quad \left(\frac{\partial}{\partial t}+\frac{u\partial}{\partial r}\right)[rv-\tfrac{1}{2}\omega(r^2-a^2)]=0,$$

$rv-\frac{1}{2}\omega(r^2-a^2)$ does not vary along the path of an electron; and since every electron must have come from the filament,

$$rv = \tfrac{1}{2}\omega(r^2-a^2). \tag{2}$$

The equation of radial motion is

$$\frac{\partial u}{\partial t}+\frac{u\,\partial u}{\partial r}-\frac{v^2}{r}=\frac{e\,\partial V}{m\,\partial r}-\omega v. \tag{3}$$

Let I be the current per cm. from the sheath to the filament in electromagnetic units. Then I is a function of t only, and

$$4\pi enu+\frac{\partial}{\partial t}\left(\frac{r\,\partial V}{\partial r}\right)=2cI. \tag{4}$$

The contribution of the vector potential to the electric force having been neglected, we have instead of the four-dimensional Poisson's equation the spatial equation

$$\frac{\partial}{\partial r}\left(\frac{r\,\partial V}{\partial r}\right)=4\pi en, \tag{5}$$

so that we have in effect assumed that the linear dimensions of the diode are small compared with the wave-length in ether corresponding to the period of the oscillations of the space charge. By going

further we should only be investigating the waves inside the diode itself, without any compensating advantage.

In steady motion, distinguished by the suffix 0, $\frac{1}{2}m(u_0^2+v^2) = eV_0$ and $2\pi e n_0 u_0 = cI_0$. Then (5) gives

$$\frac{u_0 d}{dr}\left[\frac{rd(u_0^2+v^2)}{dr}\right] = \frac{4ecI_0}{m}.$$

The substitution

$$x = \omega \int_a^r \frac{dr}{u_0}, \qquad y = \frac{m\omega^3(r^2-a^2)}{4ecI_0}, \qquad \epsilon = \frac{m\omega^3 a^2}{4ecI_0} \tag{6}$$

converts the last equation into $\frac{d}{dx}\left[\frac{\epsilon+y}{y'}\frac{d}{dx}\left(\frac{y'^2+y^2}{\epsilon+y}\right)\right] = 2$, where dashes denote differentiation with respect to x. Thus

$$2(\epsilon+y)(y''+y-x-\alpha)-y'^2-y^2 = 0, \tag{7}$$

where α is an arbitrary constant. Then

$$u_0 = \left(\frac{ecI_0}{m\omega}\right)^{\frac{1}{2}}\frac{y'}{(\epsilon+y)^{\frac{1}{2}}}, \qquad v = \left(\frac{ecI_0}{m\omega}\right)^{\frac{1}{2}}\frac{y}{(\epsilon+y)^{\frac{1}{2}}}, \qquad V_0 = \frac{cI_0(y'^2+y^2)}{2\omega(\epsilon+y)}, \tag{8}$$

and y and y' vanish when $x = 0$. The charge per cm. inside a cylinder of radius r is $-rdV_0/2dr = cI_0(x+\alpha)/\omega$. Since the electric force is a multiple of $(x+\alpha)/(\epsilon+y)^{\frac{1}{2}}$, α vanishes for saturated currents, while for unsaturated currents α is positive and I_0 is the filament emission.

There are two easy problems of steady motion.* If the filament is thin and the current saturated, y'' vanishes when $x = 0$ as well as y and y'. Equation (7) is soluble in a power series beginning with $2x^3/3$, but it is easier to transform it by writing $x = 2X$, $y = 4Y^2$, giving $YY''+Y^2 = X$, where dashes denote differentiation with respect to X. Then

$$Y = 2\left(\frac{X^3}{3}\right)^{\frac{1}{2}}\left[1-\frac{2X^2}{19}+\frac{82X^4}{3.17.19^2}-\frac{2996X^6}{3^3.11.17.19^3}+\right.$$
$$\left.+\frac{195430X^8}{3^3.17^2.19^4.163}-\frac{838076X^{10}}{3^4.11.17^2.19^5.163}+\ldots\right],$$

six terms sufficing to determine accurately the turning-point of the stream, at which Y' vanishes. At this point $X = 2{\cdot}519$, $Y = 2{\cdot}267$, or $x = 5{\cdot}038$, $y = 20{\cdot}56$. The time taken by an electron to pass from the filament to the turning-point is $2{\cdot}85/10^7 H$ seconds, and the

* Cf. S. J. Braude, *Phys. Zeits. der Sowjetunion*, 7 (1935), 565, 667.

greatest radius ρ of sheath which will catch the electrons is $15400 I_0^{\frac{1}{2}}/H^{\frac{1}{4}}$. Measurement of ρ, I_0 and H tells us whether the current is saturated or not.

If ϵ is infinitely large (plane electrodes), $y = m\omega^3 z/4\pi e c J_0$, where $J_0 = \lim(I_0/2\pi a)$ is the current per cm.^{2} and z the distance of the point from the emitting plate. Equation (7) has the finite solution $y = x - \sin x + \alpha(1-\cos x)$, and the potential at any point is

$$V_0 = \frac{8\pi^2 e c J_0^2}{m\omega^4}[(x-\sin x)^2+(1-\cos x)(1-\cos x+2\alpha x+2\alpha^2)].$$

The stream turns at $x = 2(\pi - \tan^{-1}\alpha)$, where the inverse tangent lies between 0 and $\frac{1}{2}\pi$, x falling from 2π to π as α increases from 0 to ∞; and at this point $y = 2(\pi+\alpha-\tan^{-1}\alpha)$, rising from 2π to ∞, so that the various cases of motion can be discussed completely.

Returning to the general equation (7), let $x = \xi$, $y = \eta$ be a turning-point. Then at a point $\xi + x$ near ξ,

$$y = \eta - \frac{1}{4p}\left[x^2 - \frac{2px^3}{3} - \frac{(q+s)x^4}{8} + \frac{(5q+3s)px^5}{60} + \ldots\right], \qquad (9)$$

where

$$p = \frac{\epsilon+\eta}{-2(\xi+\alpha)(\epsilon+\eta)+2\epsilon\eta+\eta^2}, \qquad q = \frac{-2(\xi+\alpha)(\epsilon+\eta)+2\epsilon\eta+\eta^2}{(\epsilon+\eta)^2}$$

$$s = \frac{4(\xi+\alpha)(\epsilon+\eta)-(\epsilon+\eta)^2+3\epsilon^2}{3(\epsilon+\eta)^2}. \qquad (10)$$

The equations apply equally to the filament, where $p = -1/2\alpha$, $q = -2\alpha/\epsilon$, $s = (4\alpha+2\epsilon)/3\epsilon$. An awkward limiting problem arises when α and ϵ are small: there is evidently one type of solution at distances from the filament small compared with a and another at distances large compared with a, depending also on how near the current is to saturation. The series is not then useful when x is finite.

2. Double streams

If the radius of the sheath exceeds the value corresponding to $y = \eta$, the stream turns back. The theory of such a stream is obtained simply from that of an outward stream. Denote electrons moving outwards by a suffix 1 and those moving inwards by a suffix 2, and let steady currents I_{10}, I_{20} be carried by them, where I_{10}, I_{20} are reckoned as positive and need not be equal since all the electrons

may graze the sheath and some only may return. Then

$$u_{10} = -u_{20} = u_0$$

and the theory is the same as before, with

$$I_0 = I_{10}+I_{20}, \qquad n_0 = n_{10}+n_{20}.$$

Some progress can be made with the theory of motions symmetrical about the axis of the filament and differing only slightly from the steady motion considered above. The independent variable r is changed to x as before, and t is written in place of ωt. Write also

$$P = \frac{\epsilon+y}{y'}, \qquad Q = \frac{\epsilon+y}{y'^2}, \qquad \theta = \frac{I_{10}-I_{20}}{I_0}. \tag{11}$$

The radial equations of motion are

$$\frac{\omega\,\partial u_1}{\partial t}+\frac{u_1\,\partial u_1}{\partial r} = -\frac{\omega\,\partial u_2}{\partial t}+\frac{u_2\,\partial u_2}{\partial r} = \frac{e\,\partial V}{m\,\partial r}+\frac{v^2}{r}-\omega v,$$

where the sign of u_2 has been changed so that it may be positive in the steady motion. Since $u_0\,\partial/\partial r = \omega\partial/\partial x$,

$$\frac{mu_0(u_1+u_2)}{2e} = \frac{\partial\Theta}{\partial x}, \qquad \frac{m(u_1^2-u_2^2)}{4e} = -\frac{\partial\Theta}{\partial t},$$

where Θ is some function of x and t, and

$$\frac{\partial V}{\partial x} = \frac{cI_0}{2\omega}\frac{d}{dx}\frac{y^2}{\epsilon+y}+\frac{\partial}{\partial x}\frac{m(u_1^2+u_2^2)}{4e}+\frac{\partial}{\partial t}\frac{mu_0(u_1-u_2)}{2e}.$$

Since $r\partial/\partial r = 2P\partial/\partial x$, the equations corresponding to (4) and (5) are

$$4\pi e(n_1u_1-n_2u_2) = -\omega\left\{-\frac{2cI}{\omega}+\frac{\partial}{\partial t}\left(\frac{2P\,\partial V}{\partial x}\right)\right\},$$

$$4\pi e(n_1+n_2)u_0 = \frac{\omega\partial}{\partial x}\left(\frac{2P\,\partial V}{\partial x}\right).$$

Solving for n_1 and n_2 and substituting in the equation

$$\partial(n_1u_1+n_2u_2)/\partial x+u_0\,\partial(n_1-n_2)/\partial t = 0,$$

we have

$$\frac{\partial}{\partial x}\left[\frac{u_0(u_1-u_2)}{u_0(u_1+u_2)}\left\{-\frac{2cI}{\omega}+\frac{\partial}{\partial t}\left(\frac{2P\,\partial V}{\partial x}\right)\right\}-\frac{2u_1u_2}{u_0(u_1+u_2)}\frac{\partial}{\partial x}\left(\frac{2P\,\partial V}{\partial x}\right)\right]+$$

$$+\frac{\partial}{\partial t}\left[\frac{2u_0^2}{u_0(u_1+u_2)}\left\{-\frac{2cI}{\omega}+\frac{\partial}{\partial t}\left(\frac{2P\,\partial V}{\partial x}\right)\right\}+\frac{u_0(u_1-u_2)}{u_0(u_1+u_2)}\frac{\partial}{\partial x}\left(\frac{2P\,\partial V}{\partial x}\right)\right] = 0.$$

Write u_0+u_1 for u_1, and similarly for u_2, V and Θ. Then

$$\Theta_0' = mu_0^2/e = cI_0/\omega Q$$

and the current from sheath to filament is θI_0+I, where I is a function of t only. Thus

$$\frac{mu_0(u_1+u_2)}{2e}=\frac{\partial\Theta}{\partial x},\qquad \frac{mu_0(u_1-u_2)}{2e}=-\frac{\partial\Theta}{\partial t}\Big/\left(1+\frac{\omega Q}{cI_0}\frac{\partial\Theta}{\partial x}\right), \tag{12}$$

$$\frac{\partial V}{\partial x}=\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial}{\partial t}\frac{mu_0(u_1-u_2)}{2e}+\frac{\partial}{\partial x}\frac{m(u_1^2+u_2^2)}{4e}, \tag{13}$$

$$\left.\begin{aligned}\frac{\partial}{\partial x}\Bigg[\frac{u_0(u_1-u_2)}{2u_0^2+u_0(u_1+u_2)}\left\{-\frac{c\theta I_0}{\omega}-\frac{cI}{\omega}+\frac{\partial}{\partial t}\left(P\frac{\partial V}{\partial x}\right)\right\}-\\ -\frac{2(u_0+u_1)(u_0+u_2)}{2u_0^2+u_0(u_1+u_2)}\left\{\frac{cI_0}{\omega}+\frac{\partial}{\partial x}\left(P\frac{\partial V}{\partial x}\right)\right\}\Bigg]+\\ +\frac{\partial}{\partial t}\Bigg[\frac{2u_0^2}{2u_0^2+u_0(u_1+u_2)}\left\{-\frac{c\theta I_0}{\omega}-\frac{cI}{\omega}+\frac{\partial}{\partial t}\left(P\frac{\partial V}{\partial x}\right)\right\}+\\ +\frac{u_0(u_1-u_2)}{2u_0^2+u_0(u_1+u_2)}\left\{\frac{cI_0}{\omega}+\frac{\partial}{\partial x}\left(P\frac{\partial V}{\partial x}\right)\right\}\Bigg]=0.\end{aligned}\right\} \tag{14}$$

No approximations have been made so far. If squares and products of $u_1,\ldots$ can be neglected, the equations admit an exponential solution in which u_1 can be replaced by $u_1 e^{i\nu t}$, where u_1 is now a small function of x and the real part is taken in the physical interpretation. In this notation I is a small constant and (12) and (13) give

$$mu_0u_1/e=\Theta'-i\nu\Theta,\quad mu_0u_2/e=\Theta'+i\nu\Theta,\quad V'=\Theta''+\nu^2\Theta. \tag{15}$$

Expanding (14) and keeping only terms of the first degree in Θ we find

$$\left.\begin{aligned}P\Theta''''+2P'\Theta'''+(P''+Q+2\nu^2P)\Theta''+(Q'-2i\nu\theta Q+2\nu^2P')\Theta'+\\ +(-i\nu\theta Q'+\nu^2P''-\nu^2Q+\nu^4P)\Theta=-ivcI/\omega\end{aligned}\right\}. \tag{16}$$

The approximations are not valid near the turning-points $x=0$ and ξ, which are considered in § 4.

3. The equation of small oscillations

The homogeneous equation obtained by putting the right-hand side of (16) zero has regular singularities at $x=0$ and ξ, with indices differing by integers. The fundamental integrals $y_1, y_2,\ldots, y_n$ of such an equation form, as is well known, a related set containing powers of logarithms, from which the corresponding integrals $z_1, z_2,\ldots, z_n$ of the adjoint equation can be found if n is not large. The application is laborious, and the following method is much simpler.

For an equation with a regular singularity at $x=0$, let ρ_μ be a member of a group of indices differing by integers, $f(\rho)=0$ the

indicial equation and $g(\rho)$ the minimum polynomial which the first coefficient g in Frobenius's theory must be taken to be in order that no subsequent coefficient may become infinite. If m is the degree of the function $F(\rho) \equiv f(\rho)\,g(\rho)$ in $\rho - \rho_\mu$, Frobenius obtained m integrals of the homogeneous equation from the function $g(x,\rho) \equiv \sum g_\nu x^{\rho+\nu}$ and its first $m-1$ derivates with respect to ρ. One more differentiation yields a particular integral of the non-homogeneous equation having x^{ρ_μ} on the right-hand side, the integral being

$$\frac{1}{F^{(m)}(\rho_\mu)}\left[\frac{\partial^m g(x,\rho)}{\partial \rho^m}\right]_{\rho=\rho_\mu}.$$

At a point $\xi+x$ near ξ,

$$\left.\begin{aligned} P &= -\frac{2}{q}\left[\frac{1}{x}+p+\frac{(4p^2+s)x}{4}+\frac{(24p^2+5q+9s)px^2}{24}+\ldots\right] \\ Q &= -\frac{2}{q}\left[-\frac{2p}{x^2}-\frac{4p^2}{x}+\frac{(-12p^2-q-2s)p}{2}+\right. \\ &\qquad \left.+\frac{(-16p^2-3q-5s)p^2x}{2}+\ldots\right] \end{aligned}\right\}, \quad (17)$$

and equation (16) becomes

$$x^4\Theta'''' + x^3P_1\Theta''' + x^2P_2\Theta'' + xP_3\Theta' + P_4\Theta = R, \quad (18)$$

where

$$P_1 = -2+2px+(2p^2+s)x^2+\ldots,$$
$$P_2 = 2-4px+\tfrac{1}{2}(-4p^2-s+4\nu^2)x^2+\ldots,$$
$$P_3 = 4px+(4i\nu\theta p-2\nu^2)x^2+\ldots, \qquad P_4 = (-4i\nu\theta p+2\nu^2)x^2+\ldots,$$
$$R = \frac{ivcIq}{2\omega}\left[x^5-px^6-\frac{sx^7}{4}+\frac{(-5q+3s)px^8}{24}+\ldots\right],$$
$$f(\rho) = (\rho-4)(\rho-3)(\rho-1)\rho, \qquad g(\rho) = (\rho-2)\rho(\rho+1)(\rho+2)(\rho+3).$$

The complementary functions of index 0, 1, 3, 4 are

$$-\tfrac{1}{3}(1-\theta^2)\nu^2p^2(x^4+\ldots)\log x+1-2px+\tfrac{1}{2}(4p^2+2i\nu\theta p-\nu^2)x^2+\ldots,$$
$$x-px^2+\tfrac{1}{12}(-3s+8i\nu\theta p)x^3+\ldots,$$
$$x^3-\tfrac{1}{2}px^4+\tfrac{1}{40}(8p^2-3s-8i\nu\theta p-8\nu^2)x^5+\ldots,$$
$$x^4-\tfrac{2}{5}px^5-\tfrac{1}{30}(3s+2i\nu\theta p+3\nu^2)x^6+\ldots,$$

and particular integrals corresponding to x^5, x^6 are

$$\tfrac{1}{40}\left[x^5-\tfrac{1}{3}px^6-\tfrac{1}{252}(16p^2+25s+8i\nu\theta p+16\nu^2)x^7+\ldots\right],$$
$$\tfrac{1}{180}\left[x^6-\tfrac{2}{7}px^7-\ldots\right].$$

The theory fails near $x = 0$ when α is zero, and the indices are then 0, 2, $1 \pm 2^{-\frac{1}{2}}i$. This case will not be considered in detail.

4. Conditions at a turning-point

If equations (15) and (16) held over the whole range from 0 to ξ, Θ and Θ' would have to vanish at each end, otherwise u_1 or u_2 would be infinite. The absence of a complementary function of index 2 then means that u_1 and u_2 vanish. The oscillation is therefore special in that the outer boundary of the cloud of electrons does not move. The equation adjoint to (16) is obtained by changing the sign of i. Hence, if ν is real, $I\int_0^\xi \Theta^* dx + I^* \int_0^\xi \Theta\, dx = 0$, or $IV^* + I^* V = 0$, where V is the difference of potential between 0 and ξ. The mass therefore acts like a self-inductance when ν is real, and cannot form part of a self-maintained circuit.

Light is thrown on the general problem by the oscillation of frequency 0, that is the varied steady motion. Let I_0 and α in § 1 receive increments δI_0 and $\delta\alpha$, and let δu_0 be the change of u_0 for a given r common to both the original and varied motions. Since $u_0 = 2ecI_0 y'/m\omega^2 r$ from (8),

$$\delta u_0 = \frac{2ec}{m\omega^2 r}(I_0\,\delta y' + y'\,\delta I_0)$$

$$= \left[\frac{ec}{m\omega I_0(\epsilon+y)}\right]^{\frac{1}{2}}\left[I_0 y''\,\delta x + I_0\frac{\partial y'}{\partial \epsilon}\delta\epsilon + I_0\frac{\partial y'}{\partial \alpha}\delta\alpha + y'\,\delta I_0\right].$$

Since $y\,\delta I_0 + I_0\,\delta y = \epsilon\,\delta I_0 + I_0\,\delta\epsilon = 0$ from (6), and

$$\delta y = y'\,\delta x + (\partial y/\partial\epsilon)\,\delta\epsilon + (\partial y/\partial\alpha)\,\delta\alpha,$$

δu_0 can be expressed in terms of δI_0 and $\delta\alpha$, and

$$\frac{mu_0\,\delta u_0}{e} = \frac{c}{\omega(\epsilon+y)}\left[\left(-yy''+y'^2+\epsilon y''\frac{\partial y}{\partial\epsilon}-\epsilon y'\frac{\partial y'}{\partial\epsilon}\right)\delta I_0 + \right.$$

$$\left. + \left(-y''\frac{\partial y}{\partial\alpha}+y'\frac{\partial y'}{\partial\alpha}\right)I_0\,\delta\alpha\right]. \quad (19)$$

The function Θ' defined by the right-hand side of (19) satisfies the equation obtained by putting $\nu = 0$ in (16). We now show that the variation process has to be modified near a turning-point. Denote the solution of equation (7) by $y = f(x, \epsilon, \alpha)$, and let T be a turning-point on the curve $y = f(x, \epsilon+\delta\epsilon, \alpha+\delta\alpha)$ near $x = \xi$, U the point on the original curve corresponding to the same $r = \rho+\delta\rho$ as T. From (6),

$$m\omega^2\rho\,\delta\rho = 2ec[(\eta - \epsilon\,\partial\eta/\partial\epsilon)\,\delta I_0 + (\partial\eta/\partial\alpha)I_0\,\delta\alpha],$$

the formula holding only when $\delta\rho$ is negative. If $\xi-\beta$ is the abscissa of U, its ordinate is $\eta-\beta^2/4p$ from (9), and $2m\omega^3\rho\,\delta\rho = -ecI_0\beta^2/p$ from (6). T is the point $[\xi-(f_{\xi\epsilon}\delta\epsilon+f_{\xi\alpha}\delta\alpha)/f_{\xi\xi}, \eta+f_\epsilon\delta\epsilon+f_\alpha\delta\alpha]$, where the arguments ξ, ϵ, α are understood in f. Thus in applying differential methods to the point U, δx is of order $\frac{1}{2}$ where $\delta\epsilon$ and $\delta\alpha$ are reckoned of order 1, so that δx^2 cannot be neglected. Carrying the work on δy and $\delta y'$ to this order, $mu_0\delta u_0/e$ receives a correction, negligible away from the turning-point, but rising to $-cI_0q\beta^2/8\omega p$ at U, the value of Θ' from (19) being also $-cI_0q\beta^2/8\omega p$ at $x=\xi$. Thus the true value of $mu_0\delta u_0/e$ at a turning-point is twice what it would be if (15) and (16) held throughout.

This result suggests that in the general nearly steady motion u_0, u_1 and u_2 are of order $\frac{1}{2}$ at a turning-point, while the displacement of the point is of order 1. Let the point correspond to $r=\rho+\sigma$, where σ is a function of t which is negative at the time considered, and to $x=\xi-\beta$. Then $2m\omega^3\rho\sigma$ is approximately equal to $-ecI_0\beta^2/p$ from (6) and (9), and $u_0+u_1 = -u_0-u_2 = d\sigma/dt$ at the turning-point. Thus u_0+u_1 and u_0+u_2 are of order 1, and

$$mu_0u_1/e = mu_0u_2/e = -cI_0q\beta^2/4\omega p$$

to order 1. Hence at the turning-point $\partial\Theta/\partial x = -cI_0/\omega Q$, and $\partial\Theta/\partial t$ vanishes in such a way that $\dfrac{\partial\Theta}{\partial t}\Big/\left(1+\dfrac{\omega Q}{cI_0}\dfrac{\partial\Theta}{\partial x}\right)$ is of order $\frac{3}{2}$. The further discussion of equations (12), (13), and (14) is difficult, and I have not been able to find the proper modification of the last of equations (15) near a turning-point. Such a modification is needed since the electric force must, in one case at least, cease to be periodic and become independent of the time. Since the true $\partial\Theta/\partial t$ vanishes at a turning-point, no error of order 1 is committed by taking the first two of equations (15) to hold over the whole range, and making Θ in equation (16) vanish when $x=\xi$. Θ' has to be doubled at that point, so that (combining two of the equations above) σ is periodic and its complex value is $2e/m\omega^2q\rho$ times the value of Θ' when $x=\xi$, as taken from (16). That the doubling is not peculiar to the varied steady motion appears from the following argument.

Consider an inverted bowl whose height at distance r from its axis of symmetry, when at rest, is $u_0=(\rho-r)^{\frac{1}{2}}$, ρ being the outer radius. (This is the law according to which the velocity u_0 in the steady motion varies near the turning-point.) Let the bowl vibrate radially

according to any law, so that r moves at time t to $r+s$, where s is a small function of r and t, and let σ be the value of s when $r = \rho$, negative at the time considered. Then at a given r common to both bowls u_0 changes to u_0+u, where u is approximately $\frac{1}{2}(\rho-r)^{-\frac{1}{2}}s$ if r is not near ρ, and $u_0 u$ is $\frac{1}{2}s$. This corresponds to the application of (16) over the whole range. At the moving edge $\rho-r = -\sigma$, $u_0 = (-\sigma)^{\frac{1}{2}}$, and $u = -(-\sigma)^{\frac{1}{2}}$, so that $u_0 u = \sigma$. There is nothing abnormal about the oscillations near the edge, but the way in which the edge becomes vertical makes methods valid at a distance from it give half the correct result. We see also that the restriction of $\delta\rho$ and σ to be negative is not essential, both difficulties arising from the attempt to apply Eulerian methods to a problem which looks more like a Lagrangian one. A little trial, however, shows that Lagrangian methods are not promising.

ELECTRICAL NOTES

By F. B. PIDDUCK (*Oxford*)

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VIII. OSCILLATIONS WITH A SINGLE-ANODE MAGNETRON*

1. Introduction

It has been shown† that a saturated current I per cm. from a thin cylindrical filament is turned back in an axial magnetic field H at a radius $15400I^{\frac{1}{2}}/H^{\frac{1}{2}}$, from which it follows that the saturation current in an apparatus like that of Okabe‡ is of the order of an ampere (not all the dimensions of the apparatus were given), while the emission was a few milliamperes. This justifies the neglect of space charge in the present note. Much that is said of a magnetron has a bearing on the theory of the short-waved oscillations maintained by a triode, since the magnetic field of one and the grid of the other are designed to turn the electrons back. It would be out of place to discuss these triode oscillations in detail, but it seems to me that the theory of Rostagni§ is based on a misconception of electrical principles.

2. Disturbed motion of electrons

Let $-e$ be the charge and m the mass of an electron, and let

$$\omega = He/mc. \tag{1}$$

Then if the filament is thin enough and if V_0 is the steady potential of the plate above the filament, the paths of the electrons are circles of radius $(2eV_0/m\omega^2)^{\frac{1}{2}}$ passing through the filament and described with angular velocity ω about the centre.‖ By the plate we mean the coaxial cylindrical sheath, whose radius is assumed to be much smaller than the wave-length of the oscillations. If a small scalar electromagnetic potential $V = f(t)$ is added to the plate, the electric field continues to affect an electron only near the time t_0 at which it leaves the filament (and of course the time at which it returns),

* A summary is given in *Nature*, 137 (1936), 945.

† See above, p. 203.

‡ K. Okabe, *Proc. Inst. Rad. Eng.* 16 (1928), 71.

§ A. Rostagni, *Atti R. Accad. Torino*, 66 (1931), 124.

‖ A. W. Hull, *Phys. Rev.* 18 (1921), 39.

so that the distance of such an electron from the filament at time t is

$$r = (2eV_0/m\omega^2)^{\frac{1}{2}}2\sin\tfrac{1}{2}(\omega t-\omega t_0)+(2e/m\omega^2V_0)^{\frac{1}{2}}\sin\tfrac{1}{2}(\omega t-\omega t_0)f(t_0).$$

Electrons are emitted continuously from the filament and form a cloud whose outer boundary contains the electrons for which $\partial r/\partial t_0 = 0$. This makes $\omega t-\omega t_0 = \pi$ in the steady motion. Let $\omega t-\omega t_0 = \pi+\theta$ in general, where θ is of the order of $f(t)$. Then θ need not be taken into account to order 1, and the outer boundary of the cloud is given by

$$r = 2(2eV_0/m\omega^2)^{\frac{1}{2}}+(2e/m\omega^2V_0)^{\frac{1}{2}}Q, \tag{2}$$

where

$$Q = f(t-\pi/\omega). \tag{3}$$

3. Calculation of current*

Let a be the radius of the filament, which we must now take to be small but finite, b the radius of the plate, the length l of either being large compared with b. Let a thin layer of electricity, of charge $-s$ per cm., be on a cylinder of radius r coaxial with the filament, and let $-f$ be the charge per cm. of the filament, V the scalar electromagnetic potential of the plate above the filament in electromagnetic units. The current from plate to filament is $i = l\,df/c\,dt$, where

$$V/c = 2f\log(r/a)+2(f+s)\log(b/r).$$

Differentiating this equation with respect to t, we have

$$i = K\,dV/dt+2cKs\,d(\log r)/dt, \tag{4}$$

where K is the capacity of the pair of cylinders in electromagnetic units. The second term may be called the convection current, though electrical influence plays a part too. It flows during the whole time that the layer is between the electrodes. There is no impulsive current just as a layer falls on the plate: so crude an idea would conflict with that principle of closed circuits which led Maxwell to his theory of displacement currents. In the magnetron, the charge leaving the filament between times t_0 and t_0+dt_0 is $-cI\,dt_0$, where I is the emission in electromagnetic units, and this gives a convection current $2cK(cI\,dt_0/l)\partial(\log r)/\partial t$. We can take r, to the first order of small quantities, from the steady motion, so that the element of convection current is $-\{2c^2KI\,\partial(\log r)/l\partial t_0\}dt_0$. Integrating, we see that, if the electrons between the plates, beginning with those which left first, extend at time t from r_1 to r_2, r_3 to r_4, and so on, the total current is

$$i = K\,dV/dt+I[\log(r_1/r_2)+\log(r_3/r_4)+\ldots]/\log(b/a). \tag{5}$$

* Cf. C. J. Bakker and G. de Vries, *Physica* 1 (1934), 1046.

It is evidently necessary for maintenance that the stream of electrons shall be interrupted, otherwise the convection current is independent of t. Though equation (5) is quite general, the calculations are complicated if an electron makes several excursions before ending on the filament or plate. I shall therefore consider for the present only the simplest problem, in which all the electrons (somewhat unphysically) are emitted normally with the same finite velocity, so that there is a sharp cut-off and those which return are caught on the filament. The theory of § 2 is unaltered since V_0 has only to be changed by a constant. I shall suppose further that $\nu = \frac{1}{2}\omega$ and that the cloud of electrons just grazes the plate once in every period. Then the boundary lags a quarter period behind the applied potential (equation (3)). Grazing produces a small gap in the return stream, and if the time of grazing is a small fraction f of the period of oscillation, but large compared with the time an electron is in the strong electric field near the filament, a convection current nearly equal to I flows for time $2f\pi/\nu$ (equation (5)). This occurs after another quarter period, so that the convection current is opposed to the potential at the time being. A cylindrical magnetron with $\nu = \frac{1}{2}\omega$ resembles in this respect a plane magnetron with $\nu = \omega$ (except that the current is more concentrated), so that equality between the time of oscillation and the time of transit of a single electron is no essential feature of maintenance. Neither condition may be quite the best.

4. Single-anode magnetron in condenser circuit

Let $\nu = \frac{1}{2}\omega$ and let the grazing be specified by the small fraction f as before. Let

$$V = \sum_{n=0}^{\infty} (A_n \cos n\nu t + B_n \sin n\nu t), \tag{6}$$

where it will appear (R and I being small) that the coefficients beyond $n = 1$ are of lower order than the fundamental, as we should expect. From equation (3),

$$Q = \sum_{n=0}^{\infty} [A_n \cos(n\nu t - \tfrac{1}{2}n\pi) + B_n \sin(n\nu t - \tfrac{1}{2}n\pi)].$$

Let the cloud graze the plate at $\nu t = \pi$. Then since Q is to be a maximum when $\nu t = \pi$,

$$-A_1 + \sum_{n=2}^{\infty} n(B_n \cos \tfrac{1}{2}n\pi - A_n \sin \tfrac{1}{2}n\pi) = 0. \tag{7}$$

If j is the current into the condenser, and if the plate potential is

kept positive by a battery of E.M.F. V_0 connected through a coil of self-inductance M,

$$L\frac{d^2j}{dt^2}+R\frac{dj}{dt}+\frac{j}{C}-\frac{dV}{dt}=0, \tag{8}$$

and $V_0 = V+M(di/dt+dj/dt)$. If M is large, dropping constants,

$$i+j=0. \tag{9}$$

From the last article,

$$i = K\frac{dV}{dt}+\begin{cases}0 & \text{in general,}\\ I & \text{for a short interval } 2f\pi/\nu \text{ near } \nu t=\frac{3}{2}\pi.\end{cases}$$

Let the last function be expanded in a Fourier's series, so that

$$i = K\frac{dV}{dt}-fI\sum_{n=0}^{\infty}(C_n\cos n\nu t+D_n\sin n\nu t), \tag{10}$$

where C_n and D_n are known numbers. From (8) and (9), equating to zero the coefficients of $\cos n\nu t$ and $\sin n\nu t$,

$$RKn\nu A_n+[LKn^2\nu^2-(K+C)/C]B_n = fI(LCn^2\nu^2-1)C_n/n\nu C,$$
$$-[LKn^2\nu^2-(K+C)/C]A_n+RKn\nu B_n = fI(LCn^2\nu^2-1)D_n/n\nu C,$$

which determine A_n and B_n. The terms in A_1 and B_1 are most important. Putting $A_1 = 0$ from (7) as a first approximation, and observing that $C_1 = 0$ and $D_1 = 2$, we have

$$LKC\nu^2 = K+C \tag{11}$$

and

$$B_1 = 2fI(LC\nu^2-1)/RK\nu^2C,$$

or

$$B_1 = 2fILC/RK(K+C). \tag{12}$$

Equation (11) gives the expected frequency of oscillation, the natural frequency of the two condensers in series apart from resistance. The condition of maintenance, that this value of B_1 makes Q a *maximum* when $\nu t = \pi$, is satisfied. Equation (12) shows that the amplitude of the oscillation maintained is proportional directly to I and inversely to R. The harmonics are of order I alone.

Many other problems can be proposed and solved: for example, ν and ω can have any ratio, the grazing can last as long as we like and be at both electrodes, and the condenser can be replaced by a pair of Lecher wires, treated approximately as having a certain self-inductance, resistance and capacity per unit length. The considerations of § 3 are of wider scope, and must be taken into account in any theory of short-waved oscillations, whatever the apparatus.

ON SOLUTIONS OF ZERO ENERGY IN CERTAIN DYNAMICAL PROBLEMS

By A. WINTNER (*Baltimore*)

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THE object of the present note is to deduce a new elementary invariant relation, valid along those solutions of certain dynamical systems which run in the sub-space $h = 0$ of the phase-space, where h is the energy constant. While the class of dynamical problems in question contains the problem of three or more bodies, the invariant relation to be obtained is not at variance with the classical investigations of Bruns, Poincaré, and Painlevé. In the problem of two bodies, $h = 0$ is the sub-space of the parabolic orbits. The invariant relation in question might be verified directly but is obtained in a more natural way by using Darboux's well-known linear transformation.*

Consider a reversible conservative dynamical system characterized by a Hamiltonian function

$$H = T - U = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}p_i p_j - U, \tag{1}$$

where the real functions

$$U = U(q_1,\ldots,q_n) \text{ and } g^{ij} = g^{ji} = g^{ji}(q_1,\ldots,q_n) \tag{2}$$

are defined and have continuous partial derivatives of the second order in an n-dimensional domain D of the $(q_1,\ldots,q_n)$-space. Let $\det(g^{ij}) \neq 0$ in D and suppose that g^{ij} is homogeneous of some degree α in $q_1,\ldots,q_n$, where α is independent of i and j. Then, if (g_{ij}) denotes the reciprocal matrix $(g^{ij})^{-1}$, the function $g_{ij} = g_{ij}(q_1,\ldots,q_n)$ is homogeneous of degree $-\alpha$. Let $U = U(q_1,\ldots,q_n)$ be homogeneous of some degree β. If, for instance, (1) is the Hamiltonian function of the problem of $k = \frac{1}{3}n$ bodies in Cartesian coordinates, then $\beta = -1$, while $\alpha = 0$, since g^{ij} is independent of $q_1,\ldots,q_n$. Consider only those solutions of the canonical equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2,\ldots, n) \tag{3}$$

along which the derivative dU/dt of $U = U(t)$ vanishes for isolated

* G. Darboux, *Comptes Rendus*, 108 (1889), 449–50. Compare also T. Levi-Civita, *Rend. d. Reale Acc. d. Lincei*, (5), 24 (1915), 61–75.

values of t, at most; and restrict the consideration of these solutions to t-intervals within which $U = U(t)$ does not vanish.

Since $(g^{ij})^{-1} = (g_{ij})$,

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\frac{dq_i}{dt}\frac{dq_j}{dt} \tag{4}$$

in view of (1) and (3). The solutions of (3) which belong to a fixed value of the energy constant $h = T-U$ are the extremals of the variation problem

$$\delta\int \{2(U+h)T\}^{\frac{1}{2}}\,dt = 0 \tag{5}$$

of Maupertuis–Jacobi, where T is to be expressed in the form (4). Since (5) is homogeneous in the sense of Weierstrass, one may replace t in (5) by any admissible parameter τ along the extremal. Choose, in particular,

$$\tau = \int^{t} U\,dt, \tag{6}$$

so that, in the case of Kepler's motion, τ is the eccentric anomaly, while, in the case of the problem of three bodies, it is the parameter of Weierstrass and Sundman. It is clear from (6) and (4) that (5) may be written in the form

$$\delta\int \left\{(U+h)\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\frac{dq_i}{d\tau}\frac{dq_j}{d\tau}\right\}^{\frac{1}{2}} d\tau = 0. \tag{7}$$

Since $U \neq 0$ by assumption, it is seen by multiplying the integrand by $U^{-1}U(=1)$ that (7) is equivalent to

$$\delta\int \{2(\mathfrak{U}+1)\mathfrak{T}\}^{\frac{1}{2}}\,d\tau = 0, \tag{8}$$

where

$$\mathfrak{U} = hU^{-1}, \qquad \mathfrak{T} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} Ug_{ij}\frac{dq_i}{d\tau}\frac{dq_j}{d\tau}. \tag{9}$$

Since (5) is known to be equivalent to

$$\delta\int L\,dt = 0, \quad \text{where} \quad L = T+U \text{ and } T-U = h,$$

comparison of (8) with (5) shows that (8) is equivalent to

$$\delta\int \mathfrak{L}\,d\tau = 0, \quad \text{where} \quad \mathfrak{L} = \mathfrak{T}+\mathfrak{U} \text{ and } \mathfrak{T}-\mathfrak{U} = 1.$$

In other words, the solutions of (3) which belong to a pre-assigned value of the energy constant h may be characterized as those solutions of the equations of motion belonging to the Lagrangian

function $\mathfrak{L} = \mathfrak{T}+\mathfrak{U}$ and the time variable τ for which the energy constant $\mathfrak{h} = \mathfrak{T}-\mathfrak{U}$ is 1. Now $(g_{ij})^{-1} = (g^{ij})$; hence $(Ug_{ij})^{-1} = (g^{ij}/U)$ in view of the assumption $U \neq 0$. Consequently, on denoting by r_i the impulse canonically conjugate to the coordinate q_i in the Lagrangian function $\mathfrak{L} = \mathfrak{T}+\mathfrak{U}$, it is seen from (9) that the Hamiltonian function $\mathfrak{H} = \mathfrak{T}-\mathfrak{U}$ which corresponds to the Lagrangian function $\mathfrak{L} = \mathfrak{T}+\mathfrak{U}$ is

$$\mathfrak{H} = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}U^{-1}r_i r_j - hU^{-1}. \tag{10}$$

It is understood that the energy constant of all those solutions of

$$\frac{dq_i}{d\tau} = \frac{\partial\mathfrak{H}}{\partial r_i}, \qquad \frac{dr_i}{d\tau} = -\frac{\partial\mathfrak{H}}{\partial q_i} \quad (i = 1, 2, \ldots, n) \tag{11}$$

which correspond to solutions of (3) is $\mathfrak{h} = \mathfrak{T}-\mathfrak{U} = 1$, so that

$$\mathfrak{H} = 1. \tag{12}$$

Now consider those solutions of (3) for which the energy constant $h = H$ is zero. Then (10) goes over into

$$\mathfrak{H} = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}U^{-1}r_i r_j, \tag{13}$$

where

$$\mathfrak{H} = \mathfrak{H}(q_1, \ldots, q_n; r_1, \ldots, r_n) \tag{14}$$

in view of (2). Since the functions (2) are supposed to be homogeneous of degree β and α respectively, the function $g^{ij}U^{-1}$ of $q_1, \ldots, q_n$ is homogeneous of degree $\alpha-\beta$. Hence, the function (14) defined by (13) is homogeneous of degree $\alpha-\beta$ and 2 in $q_1, \ldots, q_n$ and $r_1, \ldots, r_n$ respectively. Accordingly,

$$\sum_{i=1}^{n} q_i\frac{\partial\mathfrak{H}}{\partial q_i} = (\alpha-\beta)\mathfrak{H}, \qquad \sum_{i=1}^{n} r_i\frac{\partial\mathfrak{H}}{\partial r_i} = 2\mathfrak{H},$$

by Euler's identity. Consequently, from (11) and (12),

$$\sum_{i=1}^{n} q_i\frac{dr_i}{d\tau} = \beta-\alpha, \qquad \sum_{i=1}^{n} r_i\frac{dq_i}{d\tau} = 2,$$

and so

$$\frac{d}{d\tau}\sum_{i=1}^{n} q_i r_i = \beta-\alpha+2, \tag{15}$$

where α and β are constants which depend only on the dynamical system, while $q_i = q_i(\tau)$, $r_i = r_i(\tau)$ is a solution of (11) satisfying the condition (12) which represents the energy equation $H = h = 0$ of

(3). It is seen from (15) that, if $\beta-\alpha+2 \neq 0$, there exists an integration constant τ_0 such that

$$\sum_{i=1}^{n} q_i r_i = (\beta-\alpha+2)(\tau-\tau_0) \tag{16}$$

for the solutions $q_i = q_i(\tau)$, $r_i = r_i(\tau)$ just mentioned.

The bilinear identity (16) is the invariant relation announced in the introduction. Since $(g^{ij}U^{-1})^{-1} = (g_{ij}U)$, it is seen from (13) and (11) that (16) may be written in the form

$$U \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} q_i \frac{dq_j}{d\tau} = (\beta-\alpha+2)(\tau-\tau_0), \tag{17}$$

where $q_i = q_i(\tau)$; while U and g_{ij} are functions of $q_1,\ldots,q_n$ in view of (2). Finally, on introducing instead of τ the time variable t of (3), it is seen from (6) that (17) goes over into

$$\sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} q_i \frac{dq_j}{dt} = (\beta-\alpha+2) \int_{t_0}^{t} U \, d\bar{t}, \tag{18}$$

where $q_i = q_i(t)$, and so $g_{ij} = g_{ij}(t)$, $U = U(t)$.

As an application, consider those solutions of the problem of three bodies for which $c = 0$, where c denotes the length of the vector representing the constant angular momentum. The case $c = 0$ is of interest from the point of view of real singularities, since it is known that a ternary collision is impossible, unless $c = 0$. It is also known that if $c = 0$, the three paths necessarily lie in a common fixed plane. Exclude the degenerate case where the three masses are collinear for every t. Then the equations of motion may be derived from a Hamiltonian function of the form (1), where $n = 3$ and the functions g^{ij} and U of q_1, q_2, q_3 are homogeneous of degree $\alpha = 0$ and $\beta = -1$ respectively. This is seen from Whittaker's reduction* of the planar problem by choosing $c = 0$. Thus if $c = 0$ and $h = 0$, then (18) is applicable and goes over into

$$\sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} q_i \frac{dq_j}{dt} = \int_{t_0}^{t} U \, d\bar{t}, \tag{19}$$

where g_{ij} and U are, according to Whittaker, algebraic functions of q_1, q_2, q_3.

The explicit relation (19) implies a theorem which has been

* E. T. Whittaker, *Analytical Dynamics* (3rd edition, Cambridge, 1927), 353.

announced without proof by Dziobek* and, apparently, has not been verified so far in the literature. Dziobek's statement is that, if $h = 0$ and $c = 0$, the problem of three bodies may be reduced to a system of the third order. Now the planar problem of three bodies may be reduced to a system of fourth order† without using the bilinear relation (19). The latter admits, therefore, the reduction of the order from four to three, if $c = 0$ and $h = 0$, the reason being that, if $c = 0$, the solution is necessarily planar.

* O. Dziobek, *Die mathematischen Theorien der Planeten-Bewegung* (Leipzig, 1888), 70.

† Cf. E. T. Whittaker, op. cit. 353.

ON THE FOURIER COEFFICIENTS OF FUNCTIONS OF BOUNDED VARIATION

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1. It is an outstanding problem whether a continuous function, of bounded variation but not absolutely continuous, can have its nth Fourier coefficients of order $n^{-1-\alpha}$, where α is a positive constant. My chief object in the present note is to answer this question, in the affirmative. I prove, in fact,

THEOREM 1. *There exists a continuous function $f(\theta)$, non-decreasing in $(0, 2\pi)$, with $f' = 0$ p.p. (so that f is not absolutely continuous) and*

$$c_N(f^*) = O(N^{-1-c}),$$

where c is a positive constant and $f^* = f - \frac{1}{2}\theta/\pi$.†

The proof is rather long and involves some tiresome calculations. It is, therefore, worth while to observe that we can produce fairly easily an example which at least makes the result extremely plausible.

2. Suppose that n is large, that $k = k_n$ is (say) $\log n$, that I_m denotes the interval of length $1/(kn)$ centred about the point $2m\pi/n$, that

$$\phi_m(\theta) = \begin{cases} kn/(n-1) & (\theta \in I_m) \\ 0 & \text{otherwise} \end{cases} \qquad (m = 1, 2, \ldots, n-1),$$

that

$$\Phi(\theta) = \int_0^\theta \Big(\sum_{m=1}^n e^{im \log m} \phi_m(t) \Big)\, dt,$$

and that $\Phi^* = \Phi + a\theta$, where a is chosen so that $\Phi(2\pi) = \Phi(0)$. Since Φ' exists p.p. and

$$\int_0^{2\pi} |\Phi'|\, d\theta = \sum \int_0^{2\pi} \phi_m(t)\, dt = 1,$$

both $\Re\Phi$ and $\Im\Phi$ are of bounded variation, and one of them is of variation at least $\frac{1}{2}$ and at most 1. Next,

$$iN^2 c_N(\Phi^*) = N c_N(\Phi^{*\prime}) = \sum_{m=1}^N e^{im \log m} N \int_{\frac{2m\pi}{n} - \frac{1}{2kn}}^{\frac{2m\pi}{n} + \frac{1}{2kn}} k e^{-iN\theta}\, d\theta$$

$$= 2k \sin \frac{N}{2kn} \sum_{m=1}^n e^{im \log m - i2\pi Nm/n}$$

† An increasing function cannot have $f(2\pi) = f(0)$, and the nearest we can do is an increasing function less a constant multiple of θ. We might alternatively arrange for f to increase in $(0, \pi)$, and decrease in $(\pi, 2\pi)$.

$$|N^2 c_N| \leqslant A\left|\sin\frac{N}{2kn}\right| kn^{\frac{1}{2}}, \tag{1}$$

since $\sum^{n} e^{im\log m + imx}$ is $O(n^{\frac{1}{2}})$ uniformly in x.† It is easily shown that the right-hand side of (1) is $O((N\log N)^{\frac{1}{2}})$, uniformly in n.‡ Thus

$$c_N = O(N^{-\frac{3}{2}}(\log N)^{\frac{1}{2}}).$$

If the variation of the appropriate one of $\Re\Phi$ and $\Im\Phi$ took place only in a set of measure zero, we should have the example we want. Actually this much is not true, but the variation does take place only in a *small* set, $\sum I_m$ (of measure $1/k$). A heuristic passage to the limit now suggests that there should exist a continuous function, of variation between $\frac{1}{2}$ and 1, whose variation takes place only in a null set, and whose c_N, as a limit of a uniform $O(N^{-1-\alpha})$, is itself $O(N^{-1-\alpha})$. It seems, however, to be very difficult to construct a function on this basis, and my actual example is obtained differently.

3. We return now to Theorem 1. The construction of f proceeds as follows. We take a very rapidly increasing sequence of integers, $\nu_1, \nu_2, \ldots$. We take ν_1 intervals of common length $l_1 = \frac{1}{2}2\pi/\nu_1$, centred at the points $2\pi x_m^{(1)}$ $(m = 1, 2, \ldots, \nu_1)$. We assume provisionally that none of these project out of $(0, 2\pi)$. These intervals, $I_m^{(1)}$ say, we call the 'white intervals of stage 1'. (They must be allowed to overlap.) Let

$$\phi_m(\theta) = \begin{cases} 2 & (\theta \in I_m^{(1)}), \\ 0 & \text{otherwise} \end{cases}$$

$$f_1(\theta) = \int_0^{\theta} \Big(\sum_1^{\nu_1} \phi_m(t) \Big)\, dt.$$

In the simplest case in which the I_1 do not overlap, f_1' is 2 or 0 according as θ does or does not belong to some $I^{(1)}$, and the graph of f_1 consists of horizontal stretches above the 'black' intervals which separate the white $I^{(1)}$, these stretches being connected by rising straight lines (above the $I^{(1)}$). In the general case it consists of horizontal stretches above the black intervals, connected by rising polygonal lines above the coalesced white intervals. The coalesced white intervals have total measure not exceeding the sum of the lengths of the $I^{(1)}$, which is half that of the interval $(0, 2\pi)$. The variation of f_1 is 2π.

† G. H. Hardy and J. E. Littlewood, *Proc. National Acad. Sc.* (1916), 583–6. A proof is given in A. Zygmund, *Trigonometrical Series*, 116–18.

‡ Consider the cases $N > kn$ and $N \leqslant kn$ separately.

For stage 2 we take, in each typical $I^{(1)}$,† ν_2 $I^{(2)}$'s, 'white intervals of stage 2', of length $l_2 = \frac{1}{2}l_1/\nu_2$, and centred at the points

$$\xi + l_1 x_m^{(2)} \qquad (m = 1, 2, \ldots, \nu_2),$$

where ξ is the left-hand end-point of the $I^{(1)}$ in question. They are assumed provisionally not to project out of the $I^{(1)}$. [The $x_m^{(2)}$ are the same in each $I^{(1)}$; in fact, generally, $x_m^{(r)}$ is going to be a function of m and ν_r only.] There are ν_1 groups of ν_2 $I^{(2)}$'s each; the total length of the $I^{(2)}$'s is $2^{-2}2\pi$. If ϕ is a function equal to 2^2 inside a typical $I^{(2)}$ and 0 elsewhere, we take

$$f_2 = \int_0^\theta (\sum \phi(t))\, dt.$$

The graph of f_2 consists of the original horizontal stretches of f_1, new horizontal stretches across the black intervals separating coalesced $I^{(2)}$'s, and rising polygonal lines connecting *every* adjacent pair of horizontal stretches.

And so on to r stages. The black intervals of the graph of f_r persist in graphs of later stages, each f_r is a non-decreasing function of variation 2π, and the (coalesced) white intervals in which f_r increases have total measure at most $2^{-r}\, 2\pi$.

The sequence f_r converges boundedly to an $f(\theta)$, continuous, non-decreasing, and derivative vanishing *p.p.* We shall find that, when the construction is suitably made, the Fourier coefficients of $f^* = f - \frac{1}{2}\theta/\pi$ are $O(N^{-1-c})$. Since $c_N(f^*) = \lim c_N(f_r^*)$ it is enough to have

$$c_N(f_r^*) = O(N^{-1-c})$$

uniformly in r. Now (supposing for convenience of writing that $N > 0$)

$$T_r = -iN^2 c_N(f_r^*) = N c_N(f_r')$$

$$= \sum{}' N \int_{I^{(r-1)}} \phi e^{-iN\theta}\, d\theta, \text{ the sum being over all } I^{(r-1)},$$

$$= \sum{}' \sum_{m=1}^{\nu_r} N \int_{\xi + l_{r-1}x_m - \frac{1}{2}l_r}^{\xi + l_{r-1}x_m + \frac{1}{2}l_r} 2^r e^{-iN\theta}\, d\theta = \sum{}' e^{-iN\xi} 2^{r+1} \sin \tfrac{1}{2}Nl_r \sum_{m=1}^{\nu_r} e^{-iNl_{r-1}x_m},$$

i.e.

$$T_r = \sum{}' e^{-iN\xi}\, 2^{r+1} \sin \frac{t}{4\nu_r} \,.\, S,$$

where

$$S = \sum_{m=1}^{\nu_r} e^{-itx_m}, \qquad t = Nl_{r+1}.$$

† We do not subdivide coalesced $I^{(1)}$'s, but the $I^{(1)}$'s themselves.

Our task is to arrange that

$$T_r = O(N^{1-c}),$$

uniformly in r.

4. It is natural to try the uniform arrangement

$$x_m = m/\nu,$$

(where ν is written for ν_r). In this case

$$S = \sum_1^\nu e^{-itm/\nu}.$$

For values of t not near a multiple of 2π this is $O(1)$; but for exceptional values it may be ν. The upshot of this is that c_N behaves unnecessarily well for most values of N, but there is no proving what we want for the exceptional values. Indeed it would seem that some c_N can actually be as large as $1/N$, since this does definitely happen when the intervals are those of Cantor's ternary set.† The lesson is that the uniform distribution of the intervals is too rigid; and the remedy is to scatter the x_m more at random, so as to replace extremes of good and bad by a mediocre but general virtue. The x_m, however, have to satisfy a number of distinct requirements; for example, when t is in the region $\nu^{-\delta}$ to ν^δ, S has to differ from the corresponding integral $r\int_0^1 e^{-itx}\,dx$ by an error smaller than a negative power of ν. While almost any 'random' distribution would probably work, it is almost inevitable that the proof should be complicated in detail, in spite of the comparative simplicity of the leading idea.

5. We choose the x_m associated with a given ν $(=\nu_r)$ to be

$$x_m = ((\nu+m)^\alpha) = ((\nu+m)^{\frac{1}{4}}),$$

where (y) denotes the fractional part of y, and α is written (for simplicity) for $\frac{1}{4}$. The relevant properties of the x_m are collected in

LEMMA 1. *For every sufficiently small positive β there exists a positive $b = b(\beta)$ such that*

$$S = \sum_1^\nu e^{-itx_m} = O(\nu^{1-\beta}), \tag{1}$$

uniformly in

$$\nu^\beta < t \leqslant \nu^{1+\beta}; \tag{2}$$

and

$$S - \nu I = S - \nu\int_0^1 e^{-itx}\,dx = O(\nu^{1-b}), \tag{3}$$

uniformly in

$$\nu^{-\beta} < t \leqslant \nu^\beta. \tag{4}$$

† A. Rajchman, *Fundamenta Math.* 3 (1922), 286–302. See A. Zygmund, *Trigonometrical Series*, 292.

In proving these results we use (van der Corput's)

LEMMA 2.† *Suppose that $\psi(u)$ is real and $\psi'(u)$ monotonic in $a \leqslant u \leqslant b$, and that $|\psi''| \geqslant \rho > 0$. Then*

$$\text{(i)} \qquad \Big|\sum_{a\leqslant m\leqslant b} e^{2\pi i\psi(m)}\Big| \leqslant (|\psi'(b)-\psi'(a)|+2)(4\rho^{-\frac{1}{2}}+A);$$

(ii) *if further $|\psi'| \leqslant \frac{1}{2}$ in (a,b), then*

$$\Big|\sum_{a\leqslant m\leqslant b} e^{2\pi i\psi(m)} - \int_a^b e^{2\pi i\psi(u)}\,du\Big| < A;$$

where the A's are absolute constants.

6. The restatements of Lemma 1 are what we actually use: in proving it we divide the range $(\nu^{-\beta}, \nu^{1+\beta})$ differently, and prove

$$\text{(i)} \qquad S = O(\nu^{1-b}), \tag{5}$$

$$\text{uniformly in} \qquad \nu^{1-\alpha} < t \leqslant \nu^{1+\beta}; \tag{6}$$

$$\text{and (ii)} \qquad S-\nu I = O(\nu^{1-\beta}), \tag{7}$$

$$\text{uniformly in} \qquad \nu^{-\beta} < t \leqslant \nu^{1-\alpha}. \tag{8}$$

Since $I = O(\nu^{-\beta})$ in $t > \nu^{\beta}$, these results establish Lemma 1.

The range $1 \leqslant m \leqslant \nu$ divides into a number $O(\nu^{\alpha})$ of groups G of $O(\nu^{1-\alpha})$ consecutive m's each (and two imperfect groups at the ends, whose contributions are actually trivial); in each G $(\nu+m)^{\alpha}$ increases from just over an integer, k say, to just under $k+1$, the differences concerned being $O(\nu^{\alpha-1})$. Let μ_k be the first m of G (so that $\mu_{k+1}-1$ is the last), and let S_G be the contribution to S of the m's of G.

Considering now (i) and the range (6), we have

$$S_G = e^{ikl}\sum_{\mu_k}^{\mu_{k+1}-1} e^{-it(\nu+m)^{\alpha}}$$

(the fractional-part symbol being eliminated). Let

$$2\pi\psi(u) = -t(\nu+u)^{\alpha} \qquad (0 \leqslant u \leqslant \nu).$$

Since ψ' is monotonic, $|\psi'| > A$, $|\psi''| > At\nu^{\alpha-2}$, and $t\nu^{\alpha-2} < 1$ (when β is small enough), Lemma 2 (i) gives

$$|S_G| < At\nu^{\alpha-1}(t\nu^{\alpha-2})^{-\frac{1}{2}} = At^{\frac{1}{2}}\nu^{\frac{1}{2}\alpha}.$$

This is true also for the imperfect groups, and we have

$$|S| < A\nu^{\alpha}t^{\frac{1}{2}}\nu^{\frac{1}{2}\alpha} < A\nu^{\alpha+\frac{1}{2}(1+\beta)+\frac{1}{2}\alpha}.$$

The index is $1-b$ if β is small enough, which proves (i).

† See A. Zygmund, *Trigonometrical Series*, 116–18.

To prove (ii) is more troublesome. When t satisfies (8), ψ' is monotonic in $(0, \nu)$ and $|\psi| \leqslant \frac{1}{2}$. By Lemma 2 (ii)

$$\left| S_G - \int_{\mu_k}^{\mu_{k+1}} e^{ikt} e^{-il(\nu+u)^\alpha}\, du \right| < A \tag{9}$$

(the range $\mu_{k+1}-1$ to μ_{k+1} contributing $O(1)$ to the integral). Now

$$\int_{\mu_k}^{\mu_{k+1}} = \left[e^{ikt} \frac{e^{-il(\nu+u)^\alpha}}{-it\alpha} (\nu+u)^{1-\alpha} \right]_{\mu_k}^{\mu_{k+1}} - e^{ikt} \frac{1-\alpha}{-it\alpha} \int_{\mu_k}^{\mu_{k+1}} e^{-il(\nu+u)^\alpha} (\nu+u)^{-\alpha}\, du. \tag{10}$$

The second term on the right is

$$O(t^{-1}\nu^{1-2\alpha}) \int_{\mu_k}^{\mu_{k+1}} (\nu+u)^{\alpha-1}\, du = O(t^{-1}\nu^{1-2\alpha})\left[(\nu+u)^\alpha\right]_{\mu_k}^{\mu_{k+1}} = O(t^{-1}\nu^{1-2\alpha}).$$

In the first term we have

$$(\nu+\mu_k)^\alpha = k + O(\nu^{\alpha-1}), \qquad (\nu+\mu_{k+1})^\alpha = k+1+O(\nu^{\alpha-1}),$$

and the first term itself is

$$\frac{e^{ikt}}{-it\alpha}\left(e^{-il(k+1+O(\nu^{\alpha-1}))} \left(k+1+O(\nu^{\alpha-1})\right)^{\frac{1}{\alpha}-1} - e^{-il(k+O(\nu^{\alpha-1}))} \left(k+O(\nu^{\alpha-1})\right)^{\frac{1}{\alpha}-1} \right).$$

Since k varies (with varying G) between constant multiples of ν^α, we get for this, by straightforward calculation,

$$\frac{1}{-it\alpha}(e^{-it}-1)k^{\frac{1}{\alpha}-1} + \frac{1}{-it\alpha} e^{-it}\left((k+1)^{\frac{1}{\alpha}-1} - k^{\frac{1}{\alpha}-1} \right) - O(\nu^{\alpha-1}) + O\left(\frac{\nu^{\alpha-1}}{t}\right).$$

Since there are $O(\nu^\alpha)$ complete G's we have, therefore,

$$S = O(\nu^\alpha) + O\left(\frac{\nu^\alpha}{t}\right) + O(\nu^{2\alpha-1}) + O\left(\frac{\nu^{2\alpha-1}}{t}\right) + O(\nu^{1-\alpha}) + O\left(\frac{\nu^{1-\alpha}}{t}\right) + \\ + \frac{e^{-it}-1}{-it} \frac{1}{\alpha} \sum_{k_1}^{k_2} k^{\frac{1}{\alpha}-1} + O\left(\frac{1}{t} \sum_{k_1}^{k_2} \left((k+1)^{\frac{1}{\alpha}-1} - k^{\frac{1}{\alpha}-1} \right) \right),$$

where k_1 and k_2 are the first and last values of k for complete G's. The last term is

$$O\left(\frac{k_2^{\frac{1}{\alpha}-1}}{t}\right) = O\left(\frac{\nu^{1-\alpha}}{t}\right).$$

Also

$$\frac{1}{\alpha} \sum_{k_1}^{k_2} k^{\frac{1}{\alpha}-1} = k_2^{\frac{1}{\alpha}} - k_1^{\frac{1}{\alpha}} + O(\nu^{1-\alpha}) \\ = \left((\mu_{k_2}+\nu)^\alpha + O(\nu^{\alpha-1})\right)^{\frac{1}{\alpha}} - \left((\mu_{k_1}+\nu)^\alpha + O(\nu^{\alpha-1})\right)^{\frac{1}{\alpha}} + O(\nu^{1-\alpha}),$$

which reduces to $\nu + O(\nu^{1-\alpha})$ in virtue of

$$\mu_{k_1} = O(\nu^{1-\alpha}), \qquad \mu_{k_2} = \nu + O(\nu^{1-\alpha}).$$

Hence finally (picking out the worst indices)

$$S = \nu\frac{e^{-it}-1}{-it}+O(\nu^{1-\alpha})+O\left(\frac{\nu^{1-\alpha}}{t}\right).$$

When β is small this gives the result (ii), and the proof of Lemma 1 is completed.

7. We proceed to the proof of the main theorem. We suppose β an absolute constant small enough for Lemma 1 to hold, so that b is a positive absolute constant; and we have to prove that for some positive c (naturally absolute)

$$T = T_r = \sum' e^{-iN\xi}\, 2^{r+1}\sin\frac{t}{4\nu}\, S = O(N^{1-c}), \tag{1}$$

uniformly in r, where $t = Nl_{r-1}$, $S = \sum e^{-itx_m}$. We are supposing that the $I^{(r)}$ do not project out of the $I^{(r-1)}$, in order not to complicate the proof. The final correction is made in § 8.

Since ν_r increases very rapidly, we have, writing ν for ν_r, l for l_{r-1} (not l_r),

$$\nu = l^{-\omega_r} \quad (\omega_r \to \infty). \tag{2}$$

We have to consider separately the cases (i) $t > \nu^{1+\beta}$, (ii) $\nu^{\beta} < t \leqslant \nu^{1+\beta}$, (iii) $\nu^{-\beta} < t \leqslant \nu^{\beta}$, (iv) $t \leqslant \nu^{-\beta}$.

In case (i)

$$\begin{aligned} |T| &< \sum' 2^{r+1}\nu = O\Big(\sum' 2^{r+1}\nu^{1+\frac{1}{2}\beta l}\Big), \quad \text{by (2)}, \\ &= O\Big(\sum' 2^{r+1}(Nl)^{(1+\frac{1}{2}\beta)(1+\beta)}l\Big) = O(N^{1-c_1})\sum' 2^{r+1}l_{r-1} = O(N^{1-c_1}),\dagger \end{aligned} \tag{3}$$

with $c_1 = \frac{1}{2}\beta/(1+\beta)$.

In case (ii) Lemma 1 gives

$$\begin{aligned} T &= O\Big(\sum' 2^{r-1}\frac{t}{\nu}\nu^{1-b}\Big) = O\Big(\sum' 2^{r-1}t\nu^{-\frac{1}{2}b}l\Big) \\ &= O\Big(\sum' 2^{r-1}t^{1-\frac{1}{2}b/(1+\beta)}l\Big) = O(N^{1-c_2})\sum' 2^{r-1}l = O(N^{1-c_2}), \end{aligned}$$

with $c_2 = \frac{1}{2}b/(1+\beta)$.

In cases (iii) and (iv) we consider not T_r, but $T_r - T_{r-1}$, and shall prove that

$$T_r - T_{r-1} = O(N^{1-c_3}l_{r-1}). \tag{4}$$

Now

$$\begin{aligned} T_{r-1} = -iN^2c_N(f_{r-1}) &= \sum' N\int_{I^{(r-1)}} 2^{r-1}e^{-iN\theta}\,d\theta \\ &= \sum' N2^{r-1}e^{-iN\xi}Il = \sum' 2^{r-1}e^{-iN\xi}tI, \end{aligned}$$

† $\sum' 2^{r-1}l_{r-1}$ is the variation of f_{r-1}, and has the value 2π.

where I is the integral in Lemma 1. Hence

$$T_r - T_{r-1} = \sum{}' e^{-iN\xi}\left(2^{r+1}\sin\frac{t}{4\nu}S - 2^{r-1}tI\right)$$

$$|T_r - T_{r-1}| < \sum{}' 2^{r-1}t\nu^{-1}|S-\nu I| + \sum{}' 2^{r+1}\left|\sin\frac{t}{4\nu} - \frac{t}{4\nu}\right|$$

$$= O\left(\sum{}' 2^{r-1}t\nu^{-b}\right) + O\left(\sum{}' 2^{r-1}t^3\nu^{-3}\right). \quad (5)$$

This result is true also in case (iv) in virtue of the crude inequality

$$|S-\nu I| = \left|\sum e^{O(1)} - \nu\int_0^1 e^{O(t)}\,d\theta\right| = O(\nu t) = O(\nu^{1-\beta}).$$

Now straightforward calculation based on (2) and $t < \nu^\beta$ gives

$$(t\nu^{-b} + t^3\nu^{-3})/N^{1-c_3}l^2 = O(1)$$

for any $c_3 < b/\beta$. Hence (4) follows from (5).

8. Summing up what we have proved so far, we have

$$T_r = O(N^{1-c}) \qquad (t > \nu^\beta), \quad (1)$$

$$T_r - T_{r-1} = O(N^{1-c}l_{r-1}) \qquad (t \leqslant \nu^\beta). \quad (2)$$

Let ρ be the greatest value of r for which $Nl_{r-1} > \nu_\rho^\beta$. If $r \leqslant \rho$, then $t > \nu^\beta$ and (1) is true. If, on the other hand, $r > \rho$, then

$$T_\rho = O(N^{1-c}),$$

and

$$T_r = T_\rho + \sum_{\rho+1}^{r}(T_r - T_{r-1})$$

$$= O(N^{1-c}) + O(N^{1-c})\sum_{\rho+1}^{r} l_{r-1}$$

by (2). Since $l_r \leqslant 2^{-r}2\pi$, this is $O(N^{1-c})$. Hence $T_r = O(N^{1-c})$ in any case, which is the desired result.

9. It remains to indicate what modifications are necessary to meet the fact that the $I^{(r)}$ will sometimes project out of the $I^{(r-1)}$. Now only $O(1)$ $I^{(r)}$'s out of each group G can do this; hence the number involved out of the ν $I^{(r)}$'s associated with a given $I^{(r-1)}$ is at most $O(\nu^\alpha)$. Call the number of the remainder ν'. We can modify the original construction, (i) by omitting at each stage these $O(\nu^\alpha)$ $I^{(r)}$'s, (ii) by altering the 'ϕ' for the ν' retained $I^{(r)}$'s from 2^r to $2^r\nu/\nu'$ (this is necessary to make f_r continuous). A little consideration shows that the c_N of the function thus constructed differs only trivially from the c_N calculated on the provisional (and slightly incorrect) assumption.

ON THE INTEGERS WHICH ARE THE TOTIENT OF A PRODUCT OF TWO PRIMES

By P. ERDŐS (*Manchester*)

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In a previous paper* I proved that for an infinity of n the number of solutions of the equation

$$n = (p-1)(q-1)(r-1) \quad (p, q, r \text{ primes})$$

is greater than $(\log n)^{c_1}$. The proof was elementary and was based on the fact that the normal number of prime factors of $(p-1)$ is $\log\log n$.† But then I was unable to prove even that the number of solutions of $n = (p-1)(q-1)$ is unbounded. In this paper I shall prove that, for an infinity of n, the number of solutions of the equation $n = (p-1)(q-1)$ is greater than $\exp\{\sqrt{(\log n)} - \epsilon\}$.

The proof will be shorter than that of (I), but it will not be elementary, and will be very similar to the argument used in my paper 'On the representation of an integer as the sum of k kth powers'.‡

Let p_1 be sufficiently large, $A = p_1 p_2 \ldots p_\lambda$ ($p_1, p_2, \ldots, p_\lambda$ consecutive primes),

$$(\log A)^2 < p_1 \leqslant (\log A p_{\lambda+1})^2 < 4(\log A)^2, \tag{1}$$

(since by Tschebischeff's theorem $p_{\lambda+1} < 2p_\lambda < A$), and $m = [e^{p_1}] + 1$. We evidently have $\exp\sqrt{(\log m)} > A$.

We estimate the number of solutions S of the congruence

$$(p-1)(q-1) \equiv 0 \pmod{A}$$

with

$$p, q \leqslant m.$$

We write $A = BC$ and denote by S_B the number of solutions of

$$(p-1)(q-1) \equiv 0 \pmod{A}, \quad p-1 \equiv 0 \pmod{B}, \quad q-1 \equiv 0 \pmod{C}$$

with

$$(q-1, B) = 1, \qquad p, q \leqslant m.$$

First we estimate S_B.

* See above, pp. 16–19. I shall refer to this paper as (I).

† P. Erdős, *Quart. J. of Math.* (Oxford), 6 (1935), 205–13.

‡ P. Erdős, *J. of London Math. Soc.* 11 (1936), 133–6.

From a result of Titchmarsh's* it follows that, if $y < A$, with the possible exception of a single y, the number of primes p such that

$$p \equiv 1 \pmod{y}, \qquad p \leqslant m$$

is greater than $\frac{1}{2}m/\{\phi(y)\log m\}$.

Hence, with the possible exception of a single B

$$[B \leqslant A < \exp\sqrt{(\log m)}],$$

the number of primes not exceeding m for which $p-1 \equiv 0 \pmod{B}$ is greater than $\frac{1}{2}m/\{\phi(B)\log m\}$. Similarly, with the possible exception of a single C, the number of primes q not exceeding m for which

$$q-1 \equiv 0 \pmod{C} \text{ and } (q-1, B) = 1$$

is greater than $$\frac{m}{2\phi(C)\log m} - \sum_{p_i|B} \pi(m, p_i C, 1),$$

where $\pi(m, p_i C, 1)$ denotes the number of primes not exceeding m and congruent to unity to modulus $p_i C$. But, since $p_i C < A < m^{\epsilon}$, it follows from Brun's method that

$$\pi(m, p_i C, 1) < \frac{c_1 m}{\phi(p_i C)\log m} \quad (c\text{'s denoting absolute constants}).$$

Hence $$\sum_{p_i|B} \pi(m, p_i C, 1) < \sum_{p_i|B} \frac{c_1 m}{\phi(p_i C)\log m}$$

$$< \frac{c_1 \lambda m}{\phi(C)\log m(p_1-1)} < \frac{c_2 \lambda m}{\phi(C)(\log A)^2 \log m},$$

but, from (1), $(\log A)^{2\lambda} < A$ and so $\lambda < \log A$. Thus

$$\sum_{p_i|B} \pi(m, p_i C, 1) < \frac{c_2 m}{\phi(C)\log A \log m} < \frac{m}{4\phi(C)\log m}.$$

Hence the number of primes q less than m and such that

$$q-1 \equiv 0 \pmod{C}, \qquad (q-1, B) = 1$$

* E. C. Titchmarsh, *Rendi. Circ. Mat. Palermo*, 54 (1930), 414–29. The result states (p. 424) that, if k is any integer not exceeding $\exp\sqrt{(\log x)}$ one value of k possibly excluded, and l is a fixed number prime to k, then the number of primes not exceeding x and prime to l to modulus k equals

$$\frac{1}{\phi(k)} \int_2^x \frac{du}{\log u} + Ox \exp[-c\sqrt{(\log x)}].$$

is, with one possible exception, greater than

$$\frac{m}{\phi(C)4\log m}.$$

Hence, with the possible exception of two B's,

$$S_B > \frac{m^2}{8\phi(A)(\log m)^2}.$$

We evidently have $$S > \sum_{B \nmid A} S_B,$$

since the same $q-1$ cannot occur for different B's.

Hence $$S > \frac{(2^\lambda - 2)m^2}{8\phi(A)(\log m)^2} > \frac{2^\lambda m^2}{16A(\log m)^2}.$$

But the integers of the form $(p-1)(q-1)$ with p, q less than m are evidently less than m^2. Hence we may find n ($< m^2$) a multiple of A for which the equation $n = (p-1)(q-1)$ has more than $2^\lambda/16(\log m)^2$ solutions.

But $$\lambda > \frac{\log A}{3\log\log A};$$

for all prime factors of A are less than $(\log A)^3$ since, by (1), $p_1 < 4(\log A)^2$, and the product of primes in the interval

$$(4(\log A)^2, (\log A)^3)$$

is greater than A. This fact follows from the prime-number theorem but may be deduced by elementary methods too.

Thus, since $\log m < 2p_1 < 8(\log A)^2$, we finally have

$$\frac{2^\lambda}{16(\log m)^2} > \frac{2^{\frac{\log A}{3\log\log A}}}{16.64(\log A)^4} > \exp\left(\frac{c_3 \log A}{\log\log A}\right) > \exp\{\sqrt{(\log m)} - \epsilon\}.$$

Hence the result.

By a quite different elementary method I obtained the following result. Let C be sufficiently large and $p_1 < p_2 < \ldots < p_x < n$ any set of primes such that

$$x > \frac{Cn\log\log n}{(\log n)^2},$$

then the products $(p_i - 1)(p_r - 1)$ cannot be all different.

THE REPRESENTATION OF A NUMBER AS A SUM OF FOUR 'ALMOST PROPORTIONAL' SQUARES

By E. MAITLAND WRIGHT (*Aberdeen*)

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1. In a series of papers I have discussed the possibility of representing a positive integer as a sum of s integral kth powers 'almost proportional' to assigned positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_s$. I used two distinct definitions of 'almost proportional'. Thus, if

$$n = m_1^k + m_2^k + \ldots + m_s^k$$

and

$$\left| m_i^k \sum_{i=1}^{s} \lambda_i - \lambda_i n \right| = E_i,$$

my two definitions correspond to

$$E_i = o(n), \qquad E_i = O(n^{1-\alpha+\epsilon}) \quad (i = 1, 2, \ldots, s)$$

respectively, where α is a positive function of k and s, and ϵ is any positive number. It is obvious that the second definition is the more exacting. For the first definition with $k \geqslant 3$ and $s \geqslant (k-2)2^{k-1}+5$, I used a method based on that of Hardy and Littlewood for the solution of Waring's Problem (Wright, **1**); the new idea was the appropriate use of 'weighting'. For the second definition I used the Vinogradoff-Gelbke method; the generating function is a product of trigonometric polynomials. I applied this method to the case of five or more squares (Wright, **2**, **3**) and to the case (Wright, **4**)

$$k \geqslant 3, \qquad s \geqslant (k-2)2^{k-1}+5.$$

When we turn to the case of four squares we find different results corresponding to each of the two definitions. We say that a result is true for 'almost all' n if the number of exceptional n less than any N is $O(N^{1-c})$, where c is some positive number independent of N. If we use the second definition we can prove that almost all n are the sums of four squares almost proportional to any assigned $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Here $\alpha = \frac{1}{6}$, so that

$$|m_i^2(\lambda_1+\lambda_2+\lambda_3+\lambda_4) - \lambda_i n| = O(n^{\frac{5}{6}+\epsilon}).$$

I hope to publish results of this type for four and three squares elsewhere. In the present paper we use the first definition and we find a result which is much more precise. We shall prove that every n with a sufficiently large odd factor is the sum of four squares almost

proportional (with the meaning of the first definition) to any assigned $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. We thus determine which are the exceptional n; the number of exceptions less than N is $O(\log N)$. We also show that the n for which we do not prove our result are precisely the n for which this result is untrue (Theorem 4); that is, within the limits of the first definition, our result is the best possible.

The previous results in this direction are these: Lagrange (**1**) proved that every positive integer n can be expressed in the form

$$n = m_1^2+m_2^2+m_3^2+m_4^2 \tag{1.1}$$

with every m_i a positive integer or zero; Dubois (**1**) showed that, if n is odd and greater than 41, then every m_i can be taken positive; Chowla (**1**) has proved that, if $\epsilon > 0$ and n is odd and greater than some $n_0 = n_0(\epsilon)$, then every m_i can be chosen greater than $n^{\frac{1}{2}-\epsilon}$.

For the present, we take $\gamma, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ to be arbitrary positive numbers. We use λ to denote the set of four numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; that is, $l = l(\lambda)$ implies that l depends on the four parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. We write

$$\Lambda = \lambda_1+\lambda_2+\lambda_3+\lambda_4.$$

If there is a solution of (1.1) for which

$$(1-\gamma)\frac{\lambda_i}{\Lambda} < \frac{m_i^2}{n} < (1+\gamma)\frac{\lambda_i}{\Lambda} \quad (i = 1, 2, 3, 4),$$

we say that n is the sum of four squares proportional 'within γ' to $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, or, more shortly, that n is representable.

We shall prove

THEOREM 1. *If*

(i) $\gamma, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ *are assigned positive numbers,*

(ii) $n = 2^{\xi}n'$ (n' *odd*),

(iii) $n' > n_0 = n_0(\gamma, \lambda)$,

then n *is representable.*

An interesting special case of this theorem is

THEOREM 2. *Every odd number* n *can be expressed in the form* (1.1) *with*

$$m_i = \tfrac{1}{2}n^{\frac{1}{2}}\{1+o(1)\} \quad (i = 1, 2, 3, 4).$$

By the argument of a former paper (Wright, **1**, § 1.4) we see that, without loss of generality, we may take $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ positive integers in Theorem 1. Next let us suppose that Theorem 1 is true for $\xi = 0$ and $\xi = 1$. Then we can show that the theorem is true for all $\xi > 1$.

We write $\xi = 2\xi' + \eta$, where η is zero or unity and ξ' is zero or a positive integer. Since the theorem is true for $\xi = 0$ and $\xi = 1$, we have

$$2^\eta n' = m_1'^2 + m_2'^2 + m_3'^2 + m_4'^2$$

with

$$(1-\gamma)\frac{\lambda_i}{\Lambda} < \frac{m_i'^2}{2^\eta n'} < (1+\gamma)\frac{\lambda_i}{\Lambda},$$

provided $n' > n_0$. Hence

$$n = \sum_{i=1}^{4} (2^{\xi'} m_i')^2 = \sum_{i=1}^{4} m_i^2$$

with

$$(1-\gamma)\frac{\lambda_i}{\Lambda} < \frac{m_i^2}{n} < (1-\gamma)\frac{\lambda_i}{\Lambda}.$$

Hence it is sufficient to prove

THEOREM 3. *If $\gamma > 0$, if λ_1, λ_2, λ_3, λ_4 are any assigned positive integers, if $n \not\equiv 0 \pmod 4$, and if $n > n_0 = n_0(\gamma, \lambda)$, then n is representable.*

To prove this theorem we use the Hardy-Littlewood method. This applies in its simplest form to the case of five or more squares. In the case of four squares, however, the machinery breaks down; for the refinements required here I am indebted to a memoir by Kloostermann (**1**),* certain of whose results I shall quote. I shall refer to this memoir as K.

2. Henceforth we take γ any positive number and λ_1, λ_2, λ_3, λ_4 any positive *integers*; l is any positive integer or zero. We write, for $|x| < 1$,

$$\vartheta(x) = \sum_{\nu=-\infty}^{\infty} x^{\nu^2},$$

$$\vartheta_{l,i}(x) = \left(x\frac{d}{dx}\right)^{\lambda_i l} \vartheta(x) = \sum_{\nu=-\infty}^{\infty} \nu^{2\lambda_i l} x^{\nu^2},$$

$$\{\vartheta(x)\}^4 = 1 + \sum_{n=1}^{\infty} r_0(n) x^n,$$

$$\prod_{i=1}^{4} \vartheta_{l,i}(x) = \sum_{n=1}^{\infty} r_l(n) x^n \quad (l > 0).$$

Then $r_0(n)$ denotes the number of representations of the integer n in the form (1.1), where m_i is an integer positive, negative, or zero, and representations which differ in the sign or order of the numbers m_i

* The order of the error term was later reduced by Kloostermann (**2**). The proof was simplified by Estermann (**1**). For my present purpose, however, it is best to refer to Kloostermann (**1**).

are reckoned distinct. We use $\sum_{(n)}$ to denote summation over all possible representations of a particular n in this form, so that

$$r_0(n) = \sum_{(n)} 1.$$

Further, let us write

$$\alpha_i = \Lambda m_i^2 \lambda_i^{-1} n^{-1}, \qquad P(n) = \prod_{i=1}^{4} \alpha_i^{\lambda_i},$$

and $\overline{P}(n)$ for the greatest value of $P(n)$ for the particular value of n under consideration. Hence

$$\begin{aligned} r_l(n) &= \sum_{(n)} (m_1^{\lambda_1} m_2^{\lambda_2} m_3^{\lambda_3} m_4^{\lambda_4})^{2l} \\ &= \Lambda^{-\Lambda l} n^{\Lambda l} \Big(\prod_{i=1}^{4} \lambda_i^{\lambda_i} \Big)^l \sum_{(n)} \{P(n)\}^l \\ &\leqslant \Lambda^{-\Lambda l} n^{\Lambda l} \Big(\prod_{i=1}^{4} \lambda_i^{\lambda_i} \Big)^l r_0(n) \{\overline{P}(n)\}^l. \end{aligned} \tag{2.1}$$

(L, M) is the greatest common divisor of L and M. For any particular value of $q > 1$, p denotes a positive integer such that

$$0 < p \leqslant q-1, \qquad (p, q) = 1.$$

If $q = 1$, then $p = 1$. We use $\sum_{p}$ to denote summation over all such values of p for any particular value of q. We write

$$S_{p,q,\nu} = \sum_{h=0}^{q-1} \exp\Big\{\frac{2\pi i}{q}(ph^2 + \nu h)\Big\},$$

$$S_{p,q} = S_{p,q,0}, \qquad A_1 = 1,$$

$$A_q = q^{-4} \sum_{p} (S_{p,q})^4 \exp\Big(-\frac{2\pi i p n}{q}\Big) \quad (q > 1),$$

$$S(n) = \sum_{q=1}^{\infty} A_q,$$

$$\Delta_l = \{\Gamma(\Lambda l + 2)\}^{-1} \prod_{i=1}^{4} \Gamma(\lambda_i l + \tfrac{1}{2}),$$

so that $\Delta_0 = \pi^2$. Also C is the circle

$$|x| = \exp(-1/n).$$

We write $N = [n^{\frac{1}{4}}]$ and consider the set of fractions p/q for which

$$2 \leqslant q \leqslant N, \qquad 1 \leqslant p \leqslant q-1, \qquad (p, q) = 1,$$

together with the numbers 0/1 and 1/1. Suppose these numbers arranged in ascending order and let p'/q', p/q, p''/q'' be any three

consecutive terms of the set. Then we write

$$\theta'_{p,q} = \frac{2\pi}{q(q+q'')}, \qquad \theta_{p,q} = \frac{2\pi}{q(q+q')} \quad (2 \leqslant q \leqslant N),$$

$$\theta'_{1,1} = \theta_{1,1} = \frac{2\pi}{N+1},$$

and denote by $\xi_{p,q}$ the part of C on which

$$\frac{2\pi p}{q} - \theta'_{p,q} \leqslant \psi \leqslant \frac{2\pi p}{q} + \theta_{p,q}.$$

By the theory of the Farey dissection,

$$\frac{2\pi}{q(q+N)} \leqslant \theta'_{p,q} < \frac{2\pi}{qN}, \qquad \frac{2\pi}{q(q+N)} \leqslant \theta_{p,q} < \frac{2\pi}{qN},$$

$$C = \sum_{q=1}^{N} \sum_{p} \xi_{p,q}.$$

On $\xi_{p,q}$ we write

$$x = X \exp\left(\frac{2p\pi i}{q}\right) = \exp\left(\frac{2p\pi i}{q} - \frac{1}{n} + i\theta\right),$$

where θ is the principal value of $\arg X$. Also

$$y = \frac{1}{n} - i\theta = \log\frac{1}{X},$$

and

$$\left(x\frac{d}{dx}\right)^{\lambda_l l} \equiv (-1)^{\lambda_l l}\left(\frac{d}{dy}\right)^{\lambda_l l}.$$

It is obvious that we may take γ less than unity without loss of generality; γ' is a positive number which will eventually be chosen as a function of γ; ϵ is an arbitrary positive number, to be thought of as small. The symbols A_1, A_2,..., A_5, A_ϵ, B_1, B_2 denote positive numbers. Of these, A_1,..., A_5 are absolute constants, A_ϵ depends on ϵ only, and B_1, B_2 depend on λ only. The symbol $O(\ )$ refers to the passage of n to infinity and the constant implied depends on λ, l, and ϵ only.

3 LEMMA 1. *There exists a constant A_1 such that, if $n \not\equiv 0 \pmod 4$ and if $n > A_1$, then*

$$S(n) > \frac{A_2}{\log\log n}.$$

If we put $a = b = c = d = 1$ in K, and take $n = 2^\xi n'$, where $\xi \leqslant 1$ and n' is odd, we have

(i) $\chi^{(1)} = \chi_2$, (K, 448)
(ii) $\chi_2 > A_3 2^{-\xi} > A_4$, (K, 452–3)
(iii) $\chi^{(2)} \log\log n > A_5$, (K, 449)

and hence $$S(n) = \chi^{(1)}\chi^{(2)} > \frac{A_2}{\log\log n}.$$

LEMMA 2.* *When* $l \geqslant 0$

$$|r_l(n) - \Delta_l n^{\Lambda l+1} S(n)| = O(n^{\Lambda l+\frac{1}{2}+\epsilon}).$$

By Lemma 7 of K,

$$\vartheta(x) = \phi + \Phi \quad \text{on } \xi_{p,q},$$

where $$\phi = \frac{\pi^{\frac{1}{2}} S_{p,q}}{y^{\frac{1}{2}} q}, \qquad \Phi = \frac{2\pi^{\frac{1}{2}}}{q y^{\frac{1}{2}}} \sum_{\nu=1}^{\infty} S_{p,q,\nu} \exp\left(-\frac{\pi^2 \nu^2}{q^2 y}\right).$$

Hence $$\vartheta_{l,i}(x) = \left(-\frac{d}{dy}\right)^{\lambda_i l} \vartheta(x) = \phi_i + \Phi_i,$$

where $$\phi_i = \frac{\Gamma(\lambda_i l + \frac{1}{2}) S_{p,q}}{q y^{\lambda_i l + \frac{1}{2}}}, \qquad \Phi_i = \left(-\frac{d}{dy}\right)^{\lambda_i l} \Phi.$$

We have $$\prod_{i=1}^{4} \vartheta_{l,i}(x) = \prod_{i=1}^{4} (\phi_i + \Phi_i) = U_{p,q} + V_{p,q},$$

where $$U_{p,q} = \phi_1 \phi_2 \phi_3 \phi_4$$

and $$V_{p,q} = \phi_1 \phi_2 \phi_3 \Phi_4 + \text{etc.} + \Phi_1 \Phi_2 \Phi_3 \Phi_4.$$

Hence

$$\begin{aligned} 2\pi i\, r_l(n) &= \int C\Big(\prod_{i=1}^{4} \vartheta_{l,i}(x)\Big) x^{-n-1}\, dx \\ &= \sum_{q=1}^{N} \sum_{p} \int_{\xi_{p,q}} U_{p,q} x^{-n-1}\, dx + \sum_{q=1}^{N} \sum_{p} \int_{\xi_{p,q}} V_{p,q} x^{-n-1}\, dx \\ &= J_1 + J_2. \end{aligned} \tag{3.1}$$

We write $$\theta_q = \frac{2\pi}{q(q+N)}$$

and observe that $\theta_q \leqslant \theta_{p,q}$, $\theta_q \leqslant \theta'_{p,q}$. We divide each arc $\xi_{p,q}$ into three parts, $\xi_{p,q,1}$, $\xi_{p,q,2}$, $\xi_{p,q,3}$ according as

$$-\theta'_{p,q} \leqslant \theta < -\theta_q, \quad |\theta| \leqslant \theta_q, \quad \theta_q < \theta \leqslant \theta_{p,q}.$$

For $r = 1$, 2, or 3 we write

$$J_{1,r} = \Gamma(\Lambda l + 2) \sum_{q=1}^{N} \sum_{p} S_{p,q}^4\, q^{-4} \int_{\xi_{p,q,r}} y^{-\Lambda l-2} x^{-n-1}\, dx.$$

Then

$$\begin{aligned} J_1 &= \prod_{i=1}^{4} \Gamma(\lambda_i l + \tfrac{1}{2}) \sum_{q=1}^{N} \sum_{p} S_{p,q}^4\, q^{-4} \int_{\xi_{p,q}} y^{-\Lambda l-2} x^{-n-1}\, dx \\ &= \Delta_l (J_{1,1} + J_{1,2} + J_{1,3}). \end{aligned} \tag{3.2}$$

* $r_0(n) = \pi^2 n\, S(n)$ exactly (Hardy, **1**, especially 256).

Putting $x = X\exp(2p\pi i/q)$, we have

$$J_{1,2} = \Gamma(\Lambda l+2)\sum_{q=1}^{N} A_q I_q,$$

where
$$I_q = \int_{\theta=-\theta_q}^{\theta=\theta_q} y^{-\Lambda l-2}X^{-n-1}\,dX.$$

If
$$F(X) = \sum_{\nu=1}^{\infty} \frac{\nu^{\Lambda l+1}X^{\nu}}{\Gamma(\Lambda l+2)}$$

we know (Lindelöf, **1**, 138) that on C

$$|F(X)-y^{-\Lambda l-2}| = O(1).$$

Hence
$$I_q = \int_C y^{-\Lambda l-2}X^{-n-1}\,dX + O\Big(\int_{\theta=\theta_q}^{\pi} |y^{-\Lambda l-2}X^{-n-1}||dX|\Big)$$
$$= \int_C F(X)X^{-n-1}\,dX + O(1) + O\Big(\int_{\theta_q}^{\infty} \theta^{-\Lambda l-2}\,d\theta\Big),$$

$$\Gamma(\Lambda l+2)I_q = 2\pi i n^{\Lambda l+1} + O(1) + O(\theta_q^{-\Lambda l-1}) = 2\pi i n^{\Lambda l+1} + O(q^{\Lambda l+1}N^{\Lambda l+1}).$$

Hence
$$J_{1,2} = 2\pi i n^{\Lambda l+1}S(n) + O\Big(n^{\Lambda l+1}\sum_{q=N+1}^{\infty}|A_q|\Big) + O\Big(N^{\Lambda l+1}\sum_{q=1}^{N} q^{\Lambda l+1}|A_q|\Big).$$

We now require

LEMMA 3. *If (n,q) is the highest common factor of n and q,*

$$|A_q| = O\{q^{-\frac{1}{2}+\epsilon}(n,q)^{\frac{1}{2}}\}.$$

This is Lemma 6 of K with

$$a = b = c = d = 1,\ \nu_1 = \nu_2 = \nu_3 = \nu_4 = 0, \text{ and } \mu = q-1.$$

Hence
$$\sum_{q=N+1}^{\infty}|A_q| = O\Big(\sum_{q=N+1}^{\infty} q^{-\frac{1}{2}+\epsilon}(n,q)^{\frac{1}{2}}\Big) = O\Big(\sum_{\delta|n}\delta^{\frac{1}{2}-\frac{1}{2}+\epsilon}\sum_{q_1\geqslant\frac{N+1}{\delta}} q_1^{-\frac{1}{2}+\epsilon}\Big)$$
$$= O\Big(N^{\epsilon-\frac{1}{2}}\sum_{\delta|n}\delta^{-\frac{1}{2}}\Big) = O(n^{\epsilon-\frac{1}{4}}),$$

$$\sum_{q=1}^{N} q^{\Lambda l+1}|A_q| = O\Big(\sum_{q=1}^{N} q^{\Lambda l-\frac{1}{2}+\epsilon}(n,q)^{\frac{1}{2}}\Big) = O\Big(\sum_{\delta|n}\delta^{\Lambda l+\frac{1}{2}+\epsilon}\sum_{q_1\leqslant\frac{N}{\delta}} q_1^{\Lambda l-\frac{1}{2}+\epsilon}\Big)$$
$$= O\Big(N^{\Lambda l+\frac{1}{2}+\epsilon}\sum_{\delta|n}\delta^{-\frac{1}{2}}\Big),$$

$$N^{\Lambda l+1}\sum_{q=1}^{N} q^{\Lambda l+1}|A_q| = O\Big(N^{2\Lambda l+\frac{3}{2}+\epsilon}\sum_{\delta|n}\delta^{-\frac{1}{2}}\Big) = O(n^{\Lambda l+\frac{1}{4}+\epsilon}).$$

Hence $$J_{1,2} = 2\pi i n^{\Lambda l+1} S(n) + O(n^{\Lambda l+\frac{11}{12}+\epsilon}). \tag{3.3}$$

The above work follows that of K (439, 440), fairly closely. By similar extensions of the methods of K we can prove that $J_{1,1}$ and $J_{1,3}$ are

$$O(n^{\Lambda l+\frac{1}{12}+\epsilon}).$$

Hence, by (3.2) and (3.3), we have

$$J_1 = 2\pi i \Delta_l n^{\Lambda l+1} S(n) + O(n^{\Lambda l+\frac{11}{12}+\epsilon}). \tag{3.4}$$

Again, by methods similar to those of K, we can prove that

$$J_2 = O(n^{\Lambda l+\frac{1}{2}+\epsilon}). \tag{3.5}$$

$V_{p,q}$ consists of a number of terms each of which is the product of four functions of which one or more is of the type Φ_i and the remainder are of the type ϕ_i. Let us take the particular term which corresponds to the one considered in detail in K, namely,

$$I = \sum_{q=1}^{N} \sum_{p} \int_{\xi_{p,q}} \phi_1 \phi_2 \Phi_3 \Phi_4 x^{-n-1}\, dx.$$

We define numbers C_h ($h = 0, 1, \ldots, \lambda_3 l$) by the equation

$$(-1)^{\lambda_3 l} 2\pi^{\frac{1}{2}} t^{\frac{1}{4}+\lambda_3 l} e^{\pi^2/t} \left(\frac{d}{dt}\right)^{\lambda_3 l} (t^{-\frac{1}{4}} e^{-\pi^2/t}) = \sum_{h=0}^{\lambda_3 l} C_h t^{-h}.$$

Then C_h depends on h, λ_3, and l only. Putting $t = yq^2/\nu^2$, we have

$$\Phi_3 = \sum_{h=0}^{\lambda_3 l} C_h q^{-2h-1} y^{-\lambda_3 l-h-\frac{1}{4}} \left\{ \sum_{\nu=1}^{\infty} S_{p,q,\nu} \nu^{2h} e^{-\pi^2 \nu^2 q^{-2} \gamma^{-1}} \right\}.$$

We may define numbers D_k ($k = 0, 1, \ldots, \lambda_4 l$) in terms of λ_4 and so find a similar expression for Φ_4.

Hence we have $\phi_1 \phi_2 \Phi_3 \Phi_4$ equal to a sum of $\lambda_3 \lambda_4 l^2$ terms, each of the form

$$C_h D_k q^{-2(h+k+1)} y^{-l(\lambda_3+\lambda_4)-h-k-1} \times$$

$$\times \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} S_{p,q,\mu} S_{p,q,\nu} \mu^{2h} \nu^{2k} \exp\left\{ -\frac{\pi^2(\mu^2+\nu^2)}{q^2 y} \right\}.$$

Each of these $\lambda_3 \lambda_4 l^2$ terms may be dealt with by the methods of K (442–5) and we have finally (3.5).

Lemma 2 follows at once from (3.1), (3.4), and (3.5).

4. Lemma 4. *If* $0 < \gamma' < 1$, $n \not\equiv 0 \pmod 4$, *and* $n > n_0 = n_0(\gamma', \lambda)$, *then* $\overline{P}(n) > 1-\gamma'$.

If we apply the results of Lemmas 1 and 2 to the inequality (2.1), we have

$$\{\overline{P}(n)\}^l \geqslant r_l(n)\{r_0(n)\}^{-1}\Lambda^{\Lambda l}n^{-\Lambda l}\prod_{i=1}^{4}\lambda_i^{-\lambda_i l}$$

$$= \Delta_l\Lambda^{\Lambda l}\Delta_0^{-1}\prod_{i=1}^{4}\lambda_i^{-\lambda_i l}+O(n^{-\frac{1}{6}+\epsilon}\log\log n), \tag{4.1}$$

provided only that $n \not\equiv 0 \pmod 4$.

Let us take $\epsilon = \frac{1}{36}$. Then, for any particular l there exists an $n_0 = n_0(l,\lambda)$, such that, when $n > n_0$, the absolute value of the error term in (4.1) is less than half the first term. Hence, for $n > n_0(l,\lambda)$,

$$\{\overline{P}(n)\}^l > \tfrac{1}{2}\Delta_l\Lambda^{\Lambda l}\Delta_0^{-1}\prod_{i=1}^{4}\lambda_i^{-\lambda_i l} = \{\psi(l,\lambda)\}^l \tag{4.2}$$

(say). Now, from the definition of Δ_l, we have

$$\Big|l\log\psi(l,\lambda)-\Lambda l\log\Lambda+\log\Gamma(\Lambda l+2)+$$
$$+\sum_{i=1}^{4}\{l\lambda_i\log\lambda_i-\log\Gamma(\lambda_i l+\tfrac{1}{2})\}\Big| < B_1.$$

If we use Stirling's expansion of the gamma function, this reduces to

$$|l\log\psi+\tfrac{3}{2}\log l| < B_2.$$

Hence, by choosing l sufficiently large, we can make $|\log\psi|$, and therefore $|1-\psi|$, as small as we please. Hence, given γ', we can choose $l = l_0(\gamma',\lambda)$ so that $\psi > 1-\gamma'$. Then, by (4.2), when

$$n > n_0(l_0,\lambda) = n_0(\gamma',\lambda),$$

we have $$\overline{P}(n) > 1-\gamma'.$$

LEMMA 5. *If* $\gamma < 2\Lambda$ *and*

$$P(n) > \left\{1-\left(\frac{\gamma}{2\Lambda}\right)^2\right\}^{\Lambda},$$

then $$|\alpha_i-1| < \gamma \qquad (i = 1,2,3,4).$$

This is Lemma 24 of Wright (**1**).

Since $\gamma < 1$, we may choose

$$\gamma' = 1-\left\{1-\left(\frac{\gamma}{2\Lambda}\right)^2\right\}^{\Lambda}$$

in Lemma 4, and take the particular representation of n in the form (1.1) for which $P(n) = \overline{P}(n)$. Then, by Lemmas 4 and 5,

$$|\alpha_i-1| < \gamma \quad (i = 1,2,3,4).$$

This is Theorem 3.

5. We now return to the remark that Theorem 1 is, for this definition of 'almost proportional', the best possible result of its kind; we shall prove

THEOREM 4. *For any positive A we can choose $\gamma, \lambda_1, \lambda_2, \lambda_3, \lambda_4$, so that $n = 2^\xi n'$ is not representable if $n' < A$.*

We choose $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ positive integers such that $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = 1$ and $\Lambda > 4A$, and then choose $\gamma < \Lambda^{-2}$. For any n we have two alternatives.

(i) There is a representation (1.1) of n such that

$$m_i^2 \Lambda - \lambda_i n = 0 \quad (i = 1, 2, 3, 4);$$

or (ii) for every representation (1.1) of n there is at least one value of i for which

$$|m_i^2 \Lambda - \lambda_i n| \geqslant 1.$$

In case (i), we have

$$m_i^2 = \frac{\lambda_i n}{\Lambda} \quad (i = 1, 2, 3, 4),$$

and so $\Lambda | \lambda_i n$ for all i. Let p be any prime factor of Λ and p^r the highest power of p which divides Λ. Since $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = 1$, p cannot divide each of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Hence $p^r | n$ and so $\Lambda | n$. Hence $n \geqslant \Lambda > 4A$.

In case (ii), if n is representable,

$$\gamma > \frac{\Lambda}{\lambda_i} \left| \frac{m_i^2}{n} - \frac{\lambda_i}{\Lambda} \right| = \frac{|m_i^2 \Lambda - \lambda_i n|}{\lambda_i n} \geqslant \frac{1}{\lambda_i n},$$

$$n > \frac{1}{\gamma \lambda_i} > \frac{1}{\Lambda \gamma} > \Lambda > 4A.$$

Hence we have no representable n less than $4A$.

Next, suppose that $n = 2^\xi n'$ is representable and that $\xi \geqslant 3$. Then

$$n = 2^\xi n' = m_1^2 + m_2^2 + m_3^2 + m_4^2 \equiv 0 \pmod 8.$$

But $m_i^2 \equiv 0$, 1, or 4 (mod 8), and of these 1 is clearly impossible. Hence $m_i = 2m_{i,1}$ and

$$n_1 = 2^{\xi - 2} n' = m_{1,1}^2 + m_{2,1}^2 + m_{3,1}^2 + m_{4,1}^2$$

with

$$\left| \frac{m_{i,1}^2}{n_1} - \frac{\lambda_i}{\Lambda} \right| = \left| \frac{m_i^2}{n} - \frac{\lambda_i}{\Lambda} \right| < \frac{\gamma \lambda_i}{\Lambda}.$$

Hence if n is representable so is n_1. We can repeat this process and we find that either $2n'$ or $4n'$ is representable. These are both impossible if $4n' < 4A$, that is, if $n' < A$.

6. The methods of this paper may be applied to extend the theorem proved by Kloostermann.

THEOREM 5. *Let a, b, c, d be any positive integers, let $\mathfrak{S}(n)$ denote the sum*

$$\sum_{q=1}^{\infty} q^{-4} \sum_{p} S_{ap,q} S_{bp,q} S_{cp,q} S_{dp,q} \exp\left(-\frac{2n\pi ip}{q}\right),$$

let γ, λ_1, λ_2, λ_3, λ_4 be any positive numbers, and let $n_1, n_2 \ldots$ be a sequence of positive integers such that $\mathfrak{S}(n_j) > A_\epsilon n_j^{-\epsilon}$ for every positive ϵ.

Then there exists an $n_0 = n_0(\gamma, \lambda, a, b, c, d)$ such that, if $n_j > n_0$,

$$n_j = am_1^2 + bm_2^2 + cm_3^2 + dm_4^2$$

and

$$|m_i^2(a\lambda_1 + b\lambda_2 + c\lambda_3 + d\lambda_4) - \lambda_i n_j| < \gamma n_j$$

for $i = 1, 2, 3, 4$.

To prove this we have only to alter the functions $\vartheta_{l,i}(x)$ slightly. For example, we replace $\vartheta_{l,1}(x)$ by

$$\vartheta_{l,1,a}(x) = \sum_{\nu=-\infty}^{\infty} (a\nu^2)^{\lambda_1 l} x^{a\nu^2} = \left(x\frac{d}{dx}\right)^{\lambda_1 l} \vartheta(x^a).$$

The work then proceeds substantially as before.

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